

# Noise vs computational intractability in dynamics

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## Abstract

Computation plays a key role in predicting and analyzing natural phenomena. There are two fundamental barriers to our ability to computationally understand the long-term behavior of a dynamical system that describes a natural process. The first one is unaccounted-for errors, which may make the system unpredictable beyond a very limited time horizon. This is especially true for chaotic systems, where a small change in the initial conditions may cause a dramatic shift in the trajectories. The second one is Turing-completeness. By the undecidability of the Halting Problem, the long-term prospects of a system that can simulate a Turing Machine cannot be determined computationally.

We investigate the interplay between these two forces – unaccounted-for errors and Turing-completeness. We show that the introduction of even a small amount of noise into a dynamical system is sufficient to “destroy” Turing-completeness, and to make the system’s long-term behavior computationally predictable. On a more technical level, we deal with long-term statistical properties of dynamical systems, as described by invariant measures. We show that while there are simple dynamical systems for which the invariant measures are non-computable, perturbing such systems makes the invariant measures efficiently computable. Thus, noise that makes the short term behavior of the system harder to predict, may make its long term statistical behavior computationally tractable. We also obtain some insight into the computational complexity of predicting systems affected by random noise.

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## 1 Introduction

### 1.1 Motivation and statement of the results

In this paper we investigate (non)-computability phenomena surrounding physical systems. The Church-Turing thesis asserts that any computation that can be carried out in finite time by a physical device, can be carried out by a Turing Machine. The thesis can be paraphrased in the following way: provided all the initial conditions with arbitrarily good precision, and random bits when necessary, the Turing Machine can *simulate* the physical system  $\mathcal{S}$  over any fixed period of time  $[0, T]$  for  $T < \infty$ .

In reality, however, we are often interested in more than just simulating the system for a fixed period of time. In many situations, one would like

to understand the *long term behavior properties* of  $\mathcal{S}$  when  $T \rightarrow \infty$ . Some of the important properties that fall into this category include:

1. Reachability problems: given an initial state  $x_0$  does the system  $\mathcal{S}$  ever enter a state  $x$  or a set of states  $\mathcal{X}$ ?
2. Asymptotic topological properties: given an initial state  $x_0$ , which regions of the state space are visited infinitely often by the system?
3. Asymptotic statistical properties: given an initial state  $x_0$ , does the system converge to a “steady state” distribution, and can this distribution be computed? Does the distribution depend on the initial state  $x_0$ ?

The first type of questions is studied in Control Theory [BP07] and also in Automated Verification [CGP99]. The third type of questions is commonly addressed by Ergodic Theory [Wal82, Pet83]. These questions in a variety of contexts are also studied by the mathematical field of Dynamical Systems [Mañ87]. For example, one of the celebrated achievements of the Kolmogorov-Arnold-Moser (KAM) theory and its extensions [Mos01] is in providing the understanding of question (1) above for systems of planets such as the solar system.

An important challenge one needs to address in formally analyzing the computational questions surrounding dynamical systems is the fact that some of the variables involved, such as the underlying states of  $\mathcal{S}$  may be continuous rather than discrete. These are very important formalities, which can be addressed e.g. within the framework of computable analysis [Wei00]. Other works dealing with “continuous” models of computation include [Ko91, PER89, BCSS98]. Most results, both positive and negative, that are significant in practice, usually hold true for any reasonable model of continuous computation.

Numerous results on computational properties of dynamical systems have been obtained. In general, while bounded-time simulations are usually possible, the computational outlook for the “infinite” time horizon problems is grim: the long-term behavior features of many of the interesting systems is non-computable. Notable examples include piece-wise linear maps [Moo90, AMP95], polynomial maps on the complex plane [BY06, BY07] and cellular automata [Wol02, KL09]. The proofs of these negative results, while sometimes technically involved, usually follow the same outline: (1) show that the system  $\mathcal{S}$  is “rich enough” to simulate any Turing Machine  $M$ ; (2)

show that solving the Halting Problem (or some other non-recursive problem) on  $M$  can be reduced to computing the feature  $\mathcal{F}$  in question. These proofs can be summarized in the following:

**Thesis 1.** *If the physical system is rich enough, it can simulate universal computation and therefore many of the system’s long-term features are non-computable.*

This means that while analytic methods can prove *some* long-term properties of *some* dynamical systems, for “rich enough” systems, one cannot hope to have a general closed-form *analytic* algorithm, i.e. one that is not based on simulations, that computes the properties of its long-term behavior. This fundamental phenomenon is qualitatively different from chaotic behavior, or the “butterfly effect”, which is often cited as the reason that predicting complex dynamical systems is hard beyond a very short time horizon; e.g. the weather being hard to predict a few days in advance.

A chaotic behavior means that the system is extremely sensitive to the initial conditions, thus someone with only approximate knowledge of the initial state can *predict* the system’s state only within a relatively short time horizon. This does not at all preclude one from being able to compute practically relevant *statistical* properties about the system. Returning to the weather example, the forecasters may be unable to tell us whether it will rain this Wednesday, but they can give a fairly accurate *distribution* of temperatures on September 1<sup>st</sup> next year!

On the other hand, the situation with systems as in Thesis 1 is much worse. If the system is rich enough to simulate a Turing Machine it will exhibit “*Turing Chaos*”: even its statistical properties will become non-computable, not due to precision problems with the initial conditions but due to the inherent computational hardness of the system. This even led some researchers to suggest [Wol02] that simulation is the *only* way to analyze the dynamical systems that are rich enough to simulate a universal Turing Machine.

Our goal is to better understand under which scenarios computability-theoretic barriers, rather than incomplete understanding of the system or its initial condition, preclude us from analyzing the system’s long term behavior. A notable feature, shared by several prior works on computational intractability in dynamical systems, such as [Moo90, BY06, AB01], is that the non-computability phenomenon is *not robust*: the non-computability disappears once one introduces even a small amount of noise into the system. Thus, if one believes that natural systems are inherently noisy, one

would not be able to observe such non-computability phenomena in nature. In fact, we conjecture:

**Conjecture 2.** *In finite-dimensional systems non-computable phenomena are not robust.*

Thus, we conjecture that noise actually makes long-term features of the system *easier* to predict. A notable example of a robust physical system that is Turing complete is the RAM computer. Note, however, that to implement a Turing Machine on a RAM machine one would need a machine with unlimited storage, thus such a computer, while feasible if we assume unlimited physical space, would be an infinite-dimensional system. We do not know of a way to implement a Turing Machine robustly using a finite-dimensional dynamical system.

In this paper we will focus on discrete-time dynamical systems over continuous spaces as a model for physical processes. Namely, there is a set  $X$  representing all the possible states the system  $\mathcal{S}$  can ever be in, and a function  $f : X \rightarrow X$ , representing the evolution of the system in one unit of time. In other words, if at time 0 the system is in state  $x$ , then at time  $t$  it will be in state  $f^t(x) = (f \circ f \circ \dots \circ f)(x)$  ( $t$  times).

We are interested in computing the asymptotic statistical properties of  $\mathcal{S}$  as  $t \rightarrow \infty$ . These properties are described by the *invariant measures* of the system – the possible statistical behaviors of  $f^t(x)$  once the system has converged to a “steady state” distribution. While in general there might be infinitely (even uncountably) many invariant measures, only a small portion of them are physically relevant.<sup>1</sup> A typical picture is the following: the phase space can be divided in *regions* exhibiting qualitatively different limiting behaviors. Within each region  $\mathcal{R}_i$ , for almost every initial condition  $x \in \mathcal{R}_i$ , the distribution of  $f^t(x)$  will converge to a “steady state” distribution  $\mu_i$  on  $X$ , supported on the region. We are interested in whether these distributions can be computed:

**Problem 3.** *Assume that the system  $\mathcal{S}$  has reached some stationary equilibrium distribution  $\mu$ . What is the probability  $\mu(A)$  of observing a certain event  $A$ ?*

In some sense this is the most basic question one can ask about the long-term behavior of the system  $\mathcal{S}$ . Formally, the above question corresponds

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<sup>1</sup>The problem of characterizing these *physical* measures is an important challenge in Ergodic Theory.

to the computability of the ergodic invariant measures of the system<sup>2</sup> (see Section 2). A negative answer to Problem 3 was given in [GHR11] where the authors demonstrate the existence of computable one-dimensional systems for which *every* invariant measure is non-computable. This is consistent with Thesis 1 above.

In the present paper we study Problem 3 in the presence of small random perturbations: each iteration  $f$  of the system  $\mathcal{S}$  is affected by (small) random noise. Informally, in the perturbed system  $\mathcal{S}_\varepsilon$  the state of the system jumps from  $x$  to  $f(x)$  and then disperses randomly around  $f(x)$  with distribution  $p_{f(x)}^\varepsilon(\cdot)$ . The parameter  $\varepsilon$  controls the “magnitude” of the noise, so that  $p_{f(x)}^\varepsilon(\cdot) \rightarrow f(x)$  as  $\varepsilon \rightarrow 0$ .

Our first result demonstrates that the non-computability phenomena are broken by the noise. More precisely, we show:

**Theorem A.** Let  $\mathcal{S}$  be a computable system over a compact subset  $M$  of  $\mathbb{R}^d$ . Assume  $p_{f(x)}^\varepsilon$  is uniform on the  $\varepsilon$ -ball around  $f(x)$ . Then, for almost every  $\varepsilon > 0$ , the ergodic measures of the perturbed system  $\mathcal{S}_\varepsilon$  are all computable.

The precise definition of computability of measures is given in Section 2. The assumption of uniformity on the noise is not essential, and it can be relaxed to (computable) absolute continuity. Theorem A follows from general considerations on the computability and compactness of the relevant spaces. It shows that the non-computability of invariant measures is not robust, which is consistent with the general Conjecture 2.

In addition to establishing the result on the *computability* of invariant measures in noisy systems, we obtain upper bounds on the *complexity* of computing these measures. In studying the complexity of computing the invariant measures, we restrict ourselves to the case when the system has a unique invariant measure – such systems are said to be “uniquely ergodic”.

**Theorem B.** Suppose the perturbed system  $\mathcal{S}_\varepsilon$  is uniquely ergodic and the function  $f$  is polynomial-time computable. Then there exists an algorithm  $\mathcal{A}$  that computes  $\mu$  with precision  $\alpha$  in time  $O_{\mathcal{S},\varepsilon}(\text{poly}(\frac{1}{\alpha}))$ .

Note that the upper bound is exponential in the number of precision bits we are trying to achieve. The algorithm in Theorem B can be implemented in a space-efficient way, using only  $\text{poly}(\log(1/\alpha))$  amount of space. If the noise operator has a nice analytical description, and under a mild additional assumption on  $f$ , the complexity can be improved when computing at precision below the level of the noise. For example, one could take  $p_{f(x)}^\varepsilon(\cdot)$  to

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<sup>2</sup>An ergodic measure is an invariant measure that cannot be decomposed into simpler invariant measures.

be a Gaussian around  $f(x)$ . This kind of perturbation forces the system to have a unique invariant measure, while the analytical description of the Gaussian noise can be exploited to perform a more efficient computation. We need an extra assumption that in addition to being able to compute  $f$  in polynomial time, we can also integrate its convolution with polynomial functions in polynomial time.

**Theorem C.** *Suppose the noise  $p_{f(x)}^\varepsilon(\cdot)$  is Gaussian, and  $f$  is polynomial-time integrable in the above sense. Then the computation of  $\mu$  at precision  $\delta < O(\varepsilon)$  requires time  $O_{\mathcal{S},\varepsilon}(\text{poly}(\log \frac{1}{\delta}))$ .*

As with Theorem A, we do not really need the noise to be Gaussian: any noise function with a uniformly analytic description would suffice. For the sake of simplicity, we will prove Theorem C only in the one dimensional case. The result can be easily extended to the multi-dimensional case.

Informally, Theorem C says that the behavior of the system at scales below the noise level is governed by the “micro”-analytic structure of the noise that is efficiently predictable, rather than by the “macro”-dynamic structure of  $\mathcal{S}$  that can be computationally intractable to predict. Theorem C suggests that a quantitative version of Conjecture 2 can be made: if the noise function behaves “nicely” below some precision level  $\varepsilon$ , properties of the system do not only become computable with high probability, but the computation can be carried out within error  $\delta < \varepsilon$  in time  $O_\varepsilon(\text{poly}(\log \frac{1}{\delta}))$ . We will discuss this further below.

## 1.2 Comparison with previous work

It has been previously observed that the introduction of noise may destroy non-computability in several settings [AB01, BY08]. There are two conceptual differences that distinguish our work from previous works. Firstly, we consider the statistical – rather than topological – long-term behavior of the system. We still want to be able to predict the trajectory of the system in the long run, but in a statistical sense. Secondly, we also address the computational complexity of predicting these statistical properties. In particular, Theorem C states that if the noise itself is not a source of additional computational complexity, then the “computationally simple” behavior takes over, and the system becomes polynomial-time computable below the noise level.

## 1.3 Discussion

Our quantitative results (Theorems B and C) shed light on what we think is a more general phenomenon. A given dynamical system, even if it is

Turing-complete, loses its “Turing completeness” once noise is introduced. How much computational power does it retain? To give a lower bound, one would have to show that even in the presence of noise the system is still capable of simulating a Turing Machine subject to some restrictions on its resources (e.g. *PSPACE* Turing Machines). To give an upper bound, one would have to give a generic algorithm for the noisy system, such as the ones given by Theorems B and C. For the systems we consider, informally, Theorems B and C give (when the system is “nice”) a  $PSPACE(\log 1/\varepsilon)$  upper bound on the complexity of computing the invariant measure. It is also not hard to see that  $PSPACE(\log 1/\varepsilon)$  can be reduced to the evaluation of an invariant measure of an  $\varepsilon$ -noisy system of the type we consider. Thus the computational power of these systems is  $PSPACE(\log 1/\varepsilon)$ .

This raises the general question on the computational power of noisy systems. In light of the above discussion, it is reasonable to conjecture that the computational power is given by  $PSPACE(M)$ , where  $M$  is the amount of “memory” the system has. In other words, there are  $\sim 2^M$  states that are robustly distinguishable in the presence of noise. This intuition, however, is hard to formalize for general systems, and further study is needed before such a quantitative assertion can be formulated.

## 2 Preliminaries

### 2.1 Discrete-time dynamical systems

We now attempt to give a brief description of some elementary ergodic theory in discrete time dynamical systems. For a complete treatment see for instance [Wal82, Pet83, Mañ87]. A dynamical system consists of a metric space  $X$  representing all the possible states the system can ever be, and and a map  $f : X \rightarrow X$  representing the dynamics. In principle, such a model is *deterministic* in the sense that complete knowledge of the state of the system, say  $x \in X$ , at some initial time, entirely determines the future *trajectory* of the system:  $x, f(x), f(f(x)), \dots$ . Despite of this, in many interesting situations it is impossible to predict any particular feature about any specific trajectory. This is the consequence of the famous sensitivity to initial conditions (chaotic behavior) and the impossibility to make measurements with infinite precision (approximation): two initial conditions which are very close to each other (so they are indistinguishable for the physical measurement) may diverge in time, rendering the true evolution unpredictable.

Instead, one studies the *limiting or asymptotic* behavior of the system.

A common situation is the following: the phase space can be divided in *regions* exhibiting qualitatively different limiting behaviors. Within each region, all the initial conditions give rise to a trajectory which approaches an “attractor”, on which the limiting dynamics take place (and that can be quite complicated). Thus, different initial condition within the same region may lead in long term to quite different particular behaviors, but identical in a qualitative sense. Any probability distribution supported in the region will also evolve in time, approaching a limiting *invariant* distribution, supported in the attractor, and which describes in statistical terms the dynamics of the *equilibrium* situation. Formally, a probability measure  $\mu$  is **invariant** if the probabilities of events do not change in time:  $\mu(f^{-1}A) = \mu(A)$ . An invariant measure  $\mu$  is **ergodic** if it cannot be decomposed:  $f^{-1}(A) = A$  implies  $\mu(A) = 1$  or  $\mu(A) = 0$ .

We now describe random perturbations of dynamical systems. A standard reference for this material is [Kif88].

### 2.1.1 Random perturbations

Let  $f$  be a dynamical system on a space  $M$  on which Lebesgue measure can be defined (say, a Riemannian manifold). Denote by  $P(M)$  the set of all Borel probability measures over  $M$ , with the weak convergence topology. We consider a family  $\{Q_x\}_{x \in M} \in P(M)$ . By a **random perturbation of  $f$**  we will mean a Markov Chain  $X_t$ ,  $t = 0, 1, 2, \dots$  with transition probabilities  $P(A|x) = P\{X_{t+1} \in A : X_t = x\} = Q_{f(x)}(A)$  defined for any  $x \in M$ , Borel set  $A \subset M$  and  $n \in \mathbb{N}$ . We will denote the randomly perturbed dynamics  $P(\cdot|x) = Q_{f(x)}$  by  $\mathcal{S}_\varepsilon$ . Given  $\mu \in P(M)$ , the push forward under  $\mathcal{S}_\varepsilon$  is defined by  $(\mathcal{S}_*\mu)(A) = \int_M P(A|x) d\mu$ .

**Definition 4.** A probability measure  $\mu$  on  $M$  is called an **invariant measure of the random perturbation  $\mathcal{S}_\varepsilon$  of  $f$**  if  $\mathcal{S}_*\mu = \mu$ .

We will be interested in *small random perturbations*. More precisely, we will consider the following choices for  $Q_x^\varepsilon$ :

1. In Theorems A and B we choose  $Q_x^\varepsilon$  to be uniform on the  $\varepsilon$ -ball around  $x$ . That is,  $Q_x^\varepsilon = \text{vol}|_{B(x,\varepsilon)}$  is Lebesgue measure restricted to the  $\varepsilon$ -ball about  $x$ .
2. In Theorem C we use an everywhere supported density for  $Q_x^\varepsilon = K_\varepsilon(x)$ , which is uniformly analytic. In particular, the Gaussian density of variance  $\varepsilon$  centered at  $x$  satisfies these conditions.

## 2.2 Computability of probability measures

Let us first recall some basic definitions and results established in [Gác05, HR09]. We work on the well-studied computable metric spaces (see [EH98, YMT99, Wei00, Hem02, BP03]).

**Definition 5.** A **computable metric space** is a triple  $(X, d, S)$  where:

1.  $(X, d)$  is a separable metric space,
2.  $S = \{s_i : i \in \mathbb{N}\}$  is a countable dense subset of  $X$  with a fixed numbering,
3.  $d(s_i, s_j)$  are uniformly computable real numbers.

Elements in the dense set  $S$  are called *simple* or *ideal* points. Algorithms can manipulate ideal points via their indexes, and thus the whole space can be reached by algorithmic means. Examples of spaces having natural computable metric structures are Euclidean spaces, the space of continuous functions on  $[0, 1]$  and  $L^p$  spaces w.r.t. Lebesgue measure on Euclidean spaces.

**Definition 6.** A point  $x \in X$  is said to be **computable** if there is a computable function  $\varphi : \mathbb{N} \rightarrow S$  such that

$$d(\varphi(n), x) \leq 2^{-n} \quad \text{for all } n \in \mathbb{N}.$$

Such a function  $\varphi$  will be called a **name of  $x$** .

If  $x \in X$  and  $r > 0$ , the metric ball  $B(x, r)$  is defined as  $\{y \in X : d(x, y) < r\}$ . The set  $\mathcal{B} := \{B(s, q) : s \in S, q \in \mathbb{Q}, q > 0\}$  of **simple balls**, which is a basis of the topology, has a canonical numbering  $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ . An **effective open set** is an open set  $U$  such that there is a r.e. (recursively enumerable) set  $E \subseteq \mathbb{N}$  with  $U = \bigcup_{i \in E} B_i$ . If  $X'$  is another computable metric space, a function  $f : X \rightarrow X'$  is **computable** if the sets  $f^{-1}(B'_i)$  are uniformly effectively open. Note that, by definition, a computable function must be continuous.

As an example, consider the space  $[0, 1]$ . The collection of simple balls over  $[0, 1]$  can be taken to be the intervals with dyadic rational endpoints, i.e., rational numbers with finite binary representation. Let  $\mathcal{D}$  denote the set of dyadic rational numbers. Computability of functions over  $[0, 1]$ , as defined in the paragraph above, can be characterized in terms of *oracle Turing Machines* as follows:

**Proposition 7.** *A function  $f : [0, 1] \rightarrow [0, 1]$  is computable if and only if there is an oracle Turing Machine  $M^\phi$  such that for any  $x \in [0, 1]$ , any name  $\varphi$  of  $x$ , and any  $n \in \mathbb{N}$ , on input  $n$  and oracle  $\varphi$ , will output a dyadic  $d \in \mathcal{D}$  such that  $|f(x) - d| \leq 2^{-n}$ .*

Poly-time computable functions over  $[0, 1]$  are defined as follows (see [Ko91]).

**Definition 8.**  $f : [0, 1] \rightarrow [0, 1]$  is **polynomial time computable** if there is a machine  $M$  as in the proposition above which, in addition, always halts in less than  $p(n)$  steps, for some polynomial  $p$ , regardless of what the oracle function is.

We now introduce a very general notion of computability of probability measures. When  $M$  is a computable metric space, the space  $P(M)$  of probability measures over  $M$  inherits the computable structure. The set of *simple* measures  $S_{P(M)}$  can be taken to be finite rational convex combinations of point masses supported on ideal points of  $M$ . When  $M$  is compact (which will be our case), the weak topology is compatible with the Wasserstein-Kantorovich distance:

$$W_1(\mu_1, \mu_2) = \sup_{\varphi \in 1\text{-Lip}(M)} \left| \int \varphi d\mu_1 - \int \varphi d\mu_2 \right|,$$

where  $1\text{-Lip}(M)$  denotes the space of functions with Lipschitz constant less than 1. The triple  $P(M, S_{P(M)}, W_1)$  is a computable metric space. See for instance [HR09]. This automatically gives the following notion:

**Definition 9.** A probability measure  $\mu$  is **computable** if it is a computable point of  $P(M)$ .

The definition above makes sense for any probability measure, and we will use it in Theorems A and B. One shows that for computable measures, the integral of computable functions is again computable (see [HR09]). Simple examples of computable measures are Lebesgue measure, as well as any absolutely continuous measure with a computable density function.

However, computable absolutely continuous (w.r.t. Lebesgue) measures do not necessarily have computable density functions (simply because they may not be continuous).

**Definition 10.** A probability measure  $\mu$  over  $[0, 1]$  is **polynomial time computable** if its cumulative distribution function  $F(x) = \mu([0, x])$  is polynomial time computable.

Polynomial time computability of the density function of a measure  $\mu$  does not imply poly-time computability of  $\mu$  (unless  $\mathbf{P} = \#\mathbf{P}$ , see [Ko91]). However, the situation improves under analyticity assumptions. In particular, we will rely on the following result.

**Proposition 11** ([KF88]). *Assume  $f$  is analytic and polynomial time computable on  $[0, 1]$ . Then*

- (i) *the Taylor coefficients of  $f$  form a uniformly poly-time computable sequence of real numbers and,*
- (ii) *the measure  $\mu$  with density  $f$  is polynomial time computable.*

In the proof of Theorem C, we actually show that the invariant measure  $\pi$  has a density function which is analytic and polynomial time computable.

### 3 Proof of Theorem A

#### 3.1 Outline of the proof

First observe that since  $M$  is compact and the support of any ergodic measure of  $\mathcal{S}_\varepsilon$  must contain an  $\varepsilon$ -ball, there can be only finitely many ergodic measures  $\mu_1, \mu_2, \dots, \mu_{N(\varepsilon)}$ . The algorithm to compute them first finds all regions that separate the dynamics into disjoint parts. For this we show that for almost every  $\varepsilon$ , every ergodic measure has a basin of attraction such that the support of the measure is *well* contained in the basin. More precisely, we show:

**Theorem 12.** *For all but countably many  $\varepsilon > 0$ , there exists open sets  $A_1, \dots, A_{N(\varepsilon)}$  such that for all  $i = 1, \dots, N(\varepsilon)$ :*

- (i)  *$\text{supp}(\mu_i) \subset A_i$  and,*
- (ii) *for every  $x \in A_i$ ,  $\mu_x = \mu_i$ , where  $\mu_x$  is the limiting distribution of  $\mathcal{S}_\varepsilon$  starting at  $x$ .*

This is used to construct an algorithm to find these regions, which is explained in the Section 3.2, and the proof that it terminates (Theorem 25) follows from Theorem 12.

The second part of the algorithm, uses compactness of the space of measures to find the ergodic measures within each region, by ruling out the ones which are not invariant. Here we use the fact that if a system is uniquely ergodic, then its invariant measure is computable (see [GHR11]). This result

is applied to the system  $\mathcal{S}_\varepsilon$  restricted to each of the regions (provided by the algorithm described in Section 3.2) where it is uniquely ergodic.

The algorithm thus obtained has the advantage of being simple and completely general. On the other hand, it is not well suited for a complexity analysis, because the search procedure is computationally extremely wasteful.

### 3.2 The Algorithm

*Proof of Theorem 12.* For  $\varepsilon > 0$ , let  $E(\varepsilon)$  be the set of ergodic measures of  $\mathcal{S}_\varepsilon$ . By compactness,  $E(\varepsilon) = \{\mu_1, \dots, \mu_{N(\varepsilon)}\}$  is finite. For a set  $A$ , we denote by  $B_\delta(A) = \{x \in M : d(x, A) < \delta\}$  the  $\delta$ -neighborhood of  $A$ , and by  $\overline{A}$  its closure. For simplicity, we assume  $M$  to be a connected manifold with no boundary so that, in particular

$$\overline{B_\delta(A)} = \{x \in M : d(x, A) \leq \delta\} = \overline{B_\delta(A)}.$$

It is clear that the support of any ergodic measure for  $\mathcal{S}_\varepsilon$  contains the support of at least one ergodic measure for  $\mathcal{S}_{\varepsilon-h}$ , for any  $h > 0$ . Therefore, the function  $N : \varepsilon \mapsto N(\varepsilon)$  is monotonic in  $\varepsilon$  and hence it can have at most countably many discontinuities.

Suppose  $N(\cdot)$  is constant on an interval containing  $\varepsilon$  and  $\varepsilon' > \varepsilon$ . Then, for any  $i$  we have

$$f(\text{supp}(\mu_i(\varepsilon))) \subset f(\text{supp}(\mu_i(\varepsilon')))$$

and therefore, since  $\varepsilon < \varepsilon'$ :

$$\overline{B_\varepsilon(f(\text{supp}(\mu_i(\varepsilon))))} \subset \text{int}(\overline{B_{\varepsilon'}(f(\text{supp}(\mu_i(\varepsilon'))))}).$$

Combining this observation with the following Lemma 13 shows that, if  $N(\cdot)$  is continuous at  $\varepsilon$ , then for any  $\varepsilon' > \varepsilon$  sufficiently close to  $\varepsilon$  (such that  $N(\varepsilon) = N(\varepsilon')$ ), it holds

$$\text{supp}(\mu_i(\varepsilon)) \subset \text{int}(\text{supp}(\mu_i(\varepsilon'))).$$

The sets  $A_i$  in the theorem can then be taken to be  $A_i = \text{int}(\text{supp}(\mu_i(\varepsilon')))$ , which finishes the proof of Theorem 12.  $\square$

**Lemma 13.** *For every  $i = 1, \dots, N(\varepsilon)$*

$$\overline{B_\varepsilon(f(\text{supp}(\mu_i(\varepsilon))))} = \text{supp}(\mu_i(\varepsilon)).$$

*Proof.* For  $\delta > 0$  we have that:

$$\mu(B(x, \delta)) = \int_M p(y, B(x, \delta)) d\mu(y) = \int_{\text{supp}(\mu)} \text{vol}(B(x, \delta)|B(f(y), \epsilon)) d\mu(y).$$

If  $d(x, f(\text{supp}(\mu))) > \epsilon$  then clearly there is a  $\delta > 0$  such that  $\mu(B(x, \delta)) = 0$  so that

$$\text{supp}(\mu_i(\epsilon)) \subseteq \overline{B_\epsilon(f(\text{supp}(\mu_i(\epsilon)))}.$$

On the other hand, if  $d(x, f(y)) < \epsilon$  for some  $y \in \text{supp}(\mu)$ , then for any  $\delta$  small enough we have

$$B(x, \delta) \subset B(y', \epsilon)$$

for any  $y' \in B(f(y), \delta)$ . It follows that  $\text{vol}(B(x, \delta)|B(f(s), \epsilon)) = \frac{\text{vol}(B_\delta)}{\text{vol}(B_\epsilon)} > 0$  for all  $s \in f^{-1}(B(f(y), \delta))$  and therefore

$$\int_{\text{supp}(\mu)} \text{vol}(B(x, \delta)|B(f(y), \epsilon)) d\mu(y) > \frac{\text{vol}(B_\delta)}{\text{vol}(B_\epsilon)} \mu(f^{-1}(B(f(y), \delta))) > 0$$

so that

$$B_\epsilon(f(\text{supp}(\mu_i(\epsilon)))) \subset \text{supp}(\mu_i(\epsilon)).$$

Since  $\text{supp}(\mu)$  is closed, the claim follows.  $\square$

We now set the language we will use in describing the algorithm computing the ergodic measures. Fix  $\epsilon > 0$ . Let  $\xi = \{\mathbf{a}_1, \dots, \mathbf{a}_\ell\}$  be a finite open cover of  $M$ .

**Definition 14.** For any open set  $A \subset M$  and any  $\delta > 0$  let

$$\xi_\delta^{\text{in}}(A) = \{\mathbf{a} \in \xi : \mathbf{a} \subset \bigcap_{x \in A} B_\delta(x)\}$$

denote the  $\delta$ -**inner neighborhood** of  $A$  in  $\xi$ .

Define the  $\delta$ -**inner iteration**  $f_{\text{in}} : 2^\xi \rightarrow 2^\xi$  by:

1.  $f_{\text{in}}(\emptyset) = \emptyset$
2. For all  $\mathbf{a} \in \xi$ ,  $f_{\text{in}}(\mathbf{a}) = \xi_\delta^{\text{in}}(f(\mathbf{a}))$ ,
3.  $f_{\text{in}}(\{\mathbf{a}_1, \dots, \mathbf{a}_m\}) = \bigcup_{i \leq m} f_{\text{in}}(\mathbf{a}_i)$ .

**Definition 15.** For any open set  $A \subset M$  and any  $\delta > 0$  let

$$\xi_\delta^{\text{out}}(A) = \{\mathbf{a} \in \xi : \mathbf{a} \cap B_\delta(A) \neq \emptyset\}$$

denote the  $\delta$ -**outer neighborhood** of  $A$  in  $\xi$ .

Define the  $\delta$ -**outer iteration**  $f_{\text{out}} : 2^\xi \rightarrow 2^\xi$  by:

1.  $f_{\text{out}}(\emptyset) = \emptyset$
2. For all  $\mathbf{a} \in \xi$ ,  $f_{\text{out}}(\mathbf{a}) = \xi_\delta^{\text{out}}(f(\mathbf{a}))$ ,
3.  $f_{\text{out}}(\{\mathbf{a}_1, \dots, \mathbf{a}_m\}) = \bigcup_{i \leq m} f_{\text{out}}(\mathbf{a}_i)$ .

**Definition 16.** An atom  $\mathbf{a} \in \xi$  is **inner-periodic** if

$$\mathbf{a} \in f_{\text{in}}^{|\xi|}(\mathbf{a}).$$

In the following, we chose  $\delta \leq \varepsilon$  and let  $\xi$  be a covering such that for a small interval around  $\delta$  and all  $\mathbf{a} \in \xi$ ,  $f_{\text{in}}(\mathbf{a})$  is constant and non empty.

**Definition 17.** The **inner orbit** of an atom  $\mathbf{a} \in \xi$  is defined to be

$$\mathcal{O}_{\text{in}}\{\mathbf{a}\} = \bigcup_{k \geq 0} f_{\text{in}}^k\{\mathbf{a}\}.$$

**Definition 18.** A collection of atoms of  $\xi$  is called **inner-irreducible** if all of them have the same inner orbit.

*Remark 19.* If a collection of atoms is inner-irreducible, then everyone of these atoms is inner-periodic.

**Proposition 20.** *The inner map  $f_{\text{in}}$  and outer map  $f_{\text{out}}$  are computable.*

*Proof.* By the choice of  $\delta$ , the condition  $\mathbf{a}' \subset \bigcap_{x \in \mathbf{a}} B_\delta(f(x))$  can be decided, which implies computability of  $f_{\text{in}}$ . Computability of  $f_{\text{out}}$  follows by a similar argument.  $\square$

**Proposition 21.** *For every  $\mathbf{a} \in \xi$ , we can decide whether or not  $\mathbf{a}$  is inner-periodic.*

*Proof.* Because  $f_{\text{in}}$  is computable.  $\square$

**The Algorithm.** The description of the algorithm to find the basins of attraction of the invariant measures  $\mu_i$  is as follows. First chose some cover  $\xi$  as above. Then:

1. Find all the inner-periodic atoms of  $\xi$ , and call their collection  $P$ .
2. (Inner Reduction) Here we reduce  $P$  to a maximal subset  $\xi_{irr}$  which contains only inner-periodic pieces whose inner-orbits are inner-irreducible and disjoint.

First compute the inner orbits  $\{O_1, \dots, O_{|P|}\}$ .

**Lemma 22.** *If  $O_i \cap O_j \neq \emptyset$  then there is  $k_{ij}$  such that*

$$O_{k_{ij}} \subset O_i \cap O_j.$$

*Proof.* Let  $\mathbf{a} \in O_i \cap O_j$ . Since  $\mathcal{O}_{in}(\mathbf{a})$  is finite, it must contain an inner-periodic element.  $\square$

To compute  $\xi_{irr}$  start by setting  $\xi_{irr} = P$ . Then, as long as there are  $\mathbf{a}_i, \mathbf{a}_j \in \xi_{irr}, i \neq j$  such that  $O_i \cap O_j \neq \emptyset$ , set

$$\xi_{irr} := (\xi_{irr} - \{\mathbf{a}_i, \mathbf{a}_j\}) \cup \{\mathbf{a}_{k_{ij}}\}.$$

**Lemma 23.**  *$\xi_{irr}$  contains only inner periodic pieces whose inner-orbits are inner-irreducible and disjoint. By construction, the cardinality of  $\xi_{irr}$  is maximal.*

*Proof.* At each step the cardinality of  $\xi_{irr}$  is reduced by 1, so that the procedure stops after at most  $|P| - 1$  steps. It is evident that the remaining atoms have disjoint inner-orbits. Let  $\mathbf{a} \in \xi_{irr}$  and  $\mathbf{a}_i \in \mathcal{O}_{in}(\mathbf{a})$ . If  $\mathbf{a}_i$  is inner-periodic, then it was eliminated during the procedure when compared against  $\mathbf{a}$ , which means that  $\mathbf{a} \in \mathcal{O}_{in}(\mathbf{a}_i)$ . If  $\mathbf{a}_i$  was not inner-periodic, then there is some inner-periodic element  $\mathbf{a}_j$  in  $\mathcal{O}_{in}(\mathbf{a}_i)$  which was eliminated when compared to  $\mathbf{a}$ , which implies that  $\mathbf{a} \in \mathcal{O}_{in}(\mathbf{a}_j) \subset \mathcal{O}_{in}(\mathbf{a}_i)$ . This shows that  $\mathcal{O}_{in}(\mathbf{a})$  is inner-irreducible. Let  $\mathbf{a}^* \notin \xi_{irr}$ . Then  $\mathbf{a}^*$  was eliminated in the procedure, which means that  $\mathcal{O}_{in}(\mathbf{a}^*)$  can not be disjoint from  $\xi_{irr}$ . The cardinality of  $\xi_{irr}$  is therefore maximal.  $\square$

*Remark 24.* The support of any ergodic measure contains the inner orbit of at least one element in  $\xi_{irr}$ .

3. If for all  $\mathbf{a}_i, \mathbf{a}_j$  in  $\xi_{irr}$ ,  $\mathcal{O}_{out}(\mathbf{a}_i) \cap \mathcal{O}_{out}(\mathbf{a}_j) = \emptyset$  then stop and return  $\xi_{irr}$ , otherwise refine  $\xi$  and go to (1).

**Theorem 25.** *For all but countably many  $\varepsilon$ , the above algorithm terminates and returns  $\xi_{irr}$ . Moreover, if  $O_i$  denotes the inner orbit of the  $i$ -th element of  $\xi_{irr}$ , then  $\mathcal{S}_\varepsilon$  has exactly  $|\xi_{irr}|$ -many ergodic measures, and the support of each of them contains exactly one of the  $O_i$ .*

*Proof.* By Theorem 12 we can assume that  $\varepsilon$  is such that there exist disjoint open sets  $A_1, \dots, A_{N(\varepsilon)}$  such that for all  $i = 1, \dots, N(\varepsilon)$ :

- (i)  $\text{supp}(\mu_i) \subset A_i$  and,
- (ii) for every  $x \in A_i$ ,  $\mu_x = \mu_i$ , where  $\mu_x$  is the limiting measure starting at  $x$ .

Therefore, each element of the list  $\xi_{irr}$  constructed in step 2, has an inner-orbit contained in the support of some ergodic measure. The algorithm terminates because of two facts: (i) for a cover  $\xi$  fine enough, the inner orbits of two different elements of the list  $\xi_{irr}$  must be contained in the support of two different ergodic measures. (ii) For a cover finer than the minimal gap between the supports and their basins, it is guaranteed that the outer orbits will be also disjoint.  $\square$

*Proof of Theorem A.* Use the above algorithm to construct the outer irreducible pieces. Each of them is a computable forward invariant set. The perturbed system  $\mathcal{S}_\varepsilon$  restricted to each of these pieces is computable and uniquely ergodic. The associated invariant measures are therefore computable ([GHR11]).  $\square$

## 4 Proof of Theorem B

### 4.1 Outline of the Proof

The idea of the algorithm is to exploit the mixing properties of the transition operator  $\mathcal{P}$  of the perturbed system  $\mathcal{S}_\epsilon$ . Since  $\mathcal{P}$  may not have a spectral gap, we construct a related transition operator  $\overline{\mathcal{P}}$  that has the same invariant measure as  $\mathcal{P}$  while having a spectral gap (see Lemma 28 and Proposition 29).

The algorithm then computes a finite matrix approximation  $Q$  of  $\overline{\mathcal{P}}$  with the following properties: (i)  $Q$  has a simple real eigenvalue near 1, (ii) the corresponding eigenvector  $\psi$  can be chosen to have only non negative entries and (iii) the density associated to  $\psi$  (see below) is  $L^1$ -close to the stationary distribution of  $\mathcal{P}$ .

To construct the main algorithm  $\mathcal{A}$ , to each precision parameter  $\alpha$  we associate a partition  $\zeta = \zeta(\alpha)$  of the space  $M$  into regular pieces of size  $\delta = 1/O(\text{poly}(\frac{1}{\alpha}))^{1/d}$ , where  $d$  denotes the dimension of  $M$ . On input  $\alpha$  the algorithm  $\mathcal{A}$  outputs a list  $\{w_{\mathbf{a}}\}_{\mathbf{a} \in \zeta}$  of  $O(\text{poly}(\frac{1}{\alpha}))$ -dyadic numbers, which is to be interpreted as the piece-wise constant function

$$\mathcal{A}(\alpha) = \sum_{\mathbf{a} \in \zeta} w_{\mathbf{a}} \mathbf{1}\{x \in \mathbf{a}\}.$$

For any atom  $\mathbf{a}_i \in \zeta$ , let  $c_i$  denote its center point. The algorithm works as follows:

1. Compute  $f(c_i)$  with some precision  $\epsilon$ , that we will specify later:  $f_\epsilon(c_i)$  (a  $\log(1/\epsilon)$ -dyadic number)
2. For every  $\mathbf{a}_j \neq \mathbf{a}_i$  do:
  - Compute  $d(f_\epsilon(c_j), c_j)$  with precision  $\epsilon$ :  $d_\epsilon(f_\epsilon(c_i), c_j)$  (also a  $\log(1/\epsilon)$ -dyadic number).
  - set  $p_{ij}$  to be an  $\epsilon$ -approximation of  $\frac{\text{vol}(\mathbf{a})}{\text{vol}(B_\epsilon)}$  iff

$$d_\epsilon(f_\epsilon(c_i), c_j) < \epsilon - m(\delta) - 2\epsilon - \delta$$

where  $m(\delta)$  (a polynomial in  $\delta$ ) denotes the uniform modulus of continuity of  $f$  (see Equation 5). Otherwise put  $p_{ij} = 0$  (one can assume all the previous numbers to be rational, and then the inequality can be decided). Clearly, the computation of each  $p_{ij}$  can be achieved in polynomial time in  $\log(1/\epsilon)$ .

3. Compute the unique normalized Perron-Frobenius eigenvector  $\psi$  of the  $|\zeta| \times |\zeta|$  matrix  $(p_{i,j})$ , and output the list  $\{w_{\mathbf{a}}\}$  where  $w_{\mathbf{a}} = \psi_{\mathbf{a}}$ .

The key point is that the matrix  $(p_{i,j})$  can be seen as a representation of the sub-Markov transition kernel  $P_{\zeta}^{\epsilon}(x, dy) = \hat{p}_x(y)dy$ , where

$$\hat{p}_x(y) = \sum_{i,j} p_{ij} \mathbf{1}\{x \in \mathbf{a}_i\} \mathbf{1}\{y \in \mathbf{a}_j\}.$$

Proposition 31 shows that the mass deficiency of the sub-Markov approximation  $P_{\zeta}^{\epsilon}$  is uniformly small. Furthermore, we have  $P_{\zeta}^{\epsilon} \leq P$ , and therefore Lemma 30 shows that the density associated to the above computed eigenvector  $\psi$  can be made  $\alpha$ -close to the invariant density of  $P$  by choosing  $\epsilon < O(\delta)$ .

One then computes a finite-dimensional approximation, which has a spectral gap. Moreover, this approximation is such that its invariant density is close to the invariant density of  $\mathcal{S}_{\epsilon}$ .

## 4.2 Rate of convergence

Here we essentially show that the Markov kernel  $P$  of the perturbed map  $\mathcal{S}_{\epsilon}$  has a spectral gap property. For any cover  $\xi$  of  $M$ ,

1. define

$$\check{\mathbf{a}}_i = \mathbf{a}_i \setminus \bigcup_{\mathbf{a} \in \xi \setminus \mathbf{a}_i} \mathbf{a}$$

for all  $\mathbf{a}_i \in \xi$ ,

2. define furthermore the sub-Markov matrix  $Q$  by

$$Q(\mathbf{a}_i \rightarrow \mathbf{a}_j) \equiv Q(i \rightarrow j) \equiv Q_{i,j} = \begin{cases} 0 & \text{if } \mathbf{a}_j \notin f_{\text{in}}(\mathbf{a}_i) \\ \frac{\text{vol}(\check{\mathbf{a}}_j)}{\text{vol}(B_{\epsilon})} & \text{if } \mathbf{a}_j \in f_{\text{in}}(\mathbf{a}_i) \end{cases}$$

for any two atoms, which defines a weighted oriented graph on  $\xi$ ,

3. and finally, define the numbers

$$N(\mathbf{a}_i \rightarrow \mathbf{a}_j) \equiv N(i \rightarrow j) \equiv N_{i,j} = \inf\{n \geq 1 : Q_{i,j}^n > 0\} \in \{1, 2, \dots, \infty\}$$

for any two atoms of  $\xi$ .

The standing assumption in this section is that the cover  $\xi$  of  $M$  is such that

$$\xi_{\text{irr}} = \bigcap_{\mathbf{a} \in \xi} \mathcal{O}_{\text{in}}(\mathbf{a}) \quad (1)$$

is non-empty. We will refer to  $\xi_{\text{irr}}$  as the inner irreducible part of  $\xi$ .

**Lemma 26** (Comparison lemma). *The estimate*

$$\mathcal{P}^m(x, A \cap \check{\mathbf{a}}_j) \geq \mathbf{1}\{x \in \mathbf{a}_i\} Q_{i,j}^m \text{vol}(A | \check{\mathbf{a}}_j)$$

is satisfied for all  $x \in M$ , any  $\mathbf{a}_j \in \xi$ , and all  $A \in \mathbb{B}$ . In particular, for any  $\mathbf{a}_i \in \xi$ , and any two  $\xi_0, \xi_1 \subset \xi$

$$\begin{aligned} \mathcal{P}^m(x, A) &\geq \mathbf{1}\{x \in \mathbf{a}_i\} \sum_{\mathbf{a}_j \in \xi_1} Q_{i,j}^m \text{vol}(A | \check{\mathbf{a}}_j) \\ \mathcal{P}^m(x, A) &\geq \sum_{\mathbf{a}_i \in \xi_0} \mathbf{1}\{x \in \check{\mathbf{a}}_i\} \sum_{\mathbf{a}_j \in \xi_1} Q_{i,j}^m \text{vol}(A | \check{\mathbf{a}}_j) \end{aligned}$$

hold true for all  $x \in M$ ,  $A \in \mathbb{B}$  and  $m \geq 1$ .

*Proof.* Let  $A \in \mathbb{B}$ , as well as  $\mathbf{a}_i \in \xi$  and  $x \in \mathbf{a}_i$  be arbitrary, but fixed. Then for any integer  $m \geq 1$  and any  $\mathbf{a}_j \in \xi$

$$\begin{aligned} \mathcal{P}^m(x, A \cap \check{\mathbf{a}}_j) &= \int \mathcal{P}^{m-1}(x, dx_{m-1}) \mathcal{P}(x_{m-1}, A \cap \check{\mathbf{a}}_j) \\ &\geq \sum_{\mathbf{a}_k \in \xi: \mathbf{a}_j \in f_{\text{in}}(\mathbf{a}_k)} \int_{\check{\mathbf{a}}_k} \mathcal{P}^{m-1}(x, dx_{m-1}) \mathcal{P}(x_{m-1}, A \cap \check{\mathbf{a}}_j) \\ &= \sum_{\mathbf{a}_k \in \xi: \mathbf{a}_j \in f_{\text{in}}(\mathbf{a}_k)} \mathcal{P}^{m-1}(x, \check{\mathbf{a}}_k) \frac{\text{vol}(A \cap \check{\mathbf{a}}_j)}{\text{vol}(B_\epsilon)} \\ &= \sum_{\mathbf{a}_k \in \xi} \mathcal{P}^{m-1}(x, \check{\mathbf{a}}_k) Q_{k,j} \text{vol}(A | \check{\mathbf{a}}_j) \end{aligned}$$

we obtain

$$\mathcal{P}^m(x, A \cap \check{\mathbf{a}}_j) \geq \sum_{\mathbf{a}_k \in \xi} \mathcal{P}(x, \check{\mathbf{a}}_k) Q_{k,j}^{m-1} \text{vol}(A | \check{\mathbf{a}}_j)$$

by induction. Because  $x \in \mathbf{a}_i$  and  $\mathcal{P}(x, \check{\mathbf{a}}_k) \geq Q_{i,k}$  we obtain the estimate

$$\mathcal{P}^m(x, A \cap \check{\mathbf{a}}_j) \geq Q_{i,j}^m \text{vol}(A | \check{\mathbf{a}}_j) \quad \text{for all } x \in \mathbf{a}_i, \mathbf{a}_j \in \xi$$

for all  $m \geq 1$ . □

Denote for  $x \in M$  and  $A \in \mathbb{B}$  by

$$\bar{\mathcal{P}}(x, A) = \frac{1}{N_\xi} \sum_{n=1}^{N_\xi} \mathcal{P}^n(x, A), \quad N_\xi = \max_{\mathbf{a}_j \in \xi} \max_{\mathbf{a}_i \in \xi_{\text{irr}}} N(\mathbf{a}_j \rightarrow \mathbf{a}_i) \quad (2)$$

a new Markov transition kernel on  $M$ . By choice of  $\xi_{\text{irr}}$  the number  $N_\xi$  is finite, and hence  $\bar{\mathcal{P}}(x, A)$  is a well-defined Markov transition kernel on  $M$ . Furthermore, let

$$\beta = \min_{\mathbf{a}_i \in \xi} \frac{1}{N_\xi} \sum_{n=1}^{N_\xi} \sum_{\mathbf{a}_j \in \xi_{\text{irr}}} Q_{i,j}^n, \quad 0 < \beta \leq 1, \quad (3)$$

where the fact that  $\beta > 0$  is shown in the following lemma.

**Lemma 27** (Lower bound on  $\beta$ ). *The following (rather pessimistic) bound on  $\beta$*

$$\beta \geq \frac{\#\xi_{\text{irr}}}{N_\xi} \left[ \min_{\mathbf{a} \in \xi} \frac{\text{vol}(\check{\mathbf{a}})}{\text{vol}(B_\epsilon)} \right]^{N_\xi}$$

holds, and shows in particular that  $\beta > 0$ .

*Proof.* From its definition in (3) we have

$$\beta = \min_{\mathbf{a}_i \in \xi} \frac{1}{N_\xi} \sum_{n=1}^{N_\xi} \sum_{\mathbf{a}_j \in \xi_{\text{irr}}} Q_{i,j}^n \geq \frac{1}{N_\xi} \min_{\mathbf{a}_i \in \xi} \sum_{\mathbf{a}_j \in \xi_{\text{irr}}} Q_{i,j}^{N_\xi}.$$

Furthermore, due to the lower bound

$$Q_{i,j} \geq \begin{cases} 0 & \text{if } \mathbf{a}_j \notin f_{\text{in}}(\mathbf{a}_i) \\ q & \text{if } \mathbf{a}_j \in f_{\text{in}}(\mathbf{a}_i) \end{cases}, \quad q = \min_{\mathbf{a} \in \xi} \frac{\text{vol}(\check{\mathbf{a}})}{\text{vol}(B_\epsilon)}$$

the above can be further estimated from below by

$$\beta \geq \frac{1}{N_\xi} \min_{\mathbf{a}_i \in \xi} \sum_{\mathbf{a}_j \in \xi_{\text{irr}}} q^{N_\xi} \geq \frac{1}{N_\xi} \min_{\mathbf{a}_i \in \xi} \sum_{\mathbf{a}_j \in \xi_{\text{irr}}} q^{N_\xi} \geq \frac{q^{N_\xi}}{N_\xi} \#\xi_{\text{irr}}.$$

□

**Lemma 28** (Doebelin condition for  $\bar{\mathcal{P}}$ ). *There exists a probability measure  $\varphi$  on  $M$  such that  $\inf_{x \in M} \bar{\mathcal{P}}(x, A) \geq \beta \varphi(A)$  holds for all  $A \in \mathbb{B}$ .*

*Proof.* By Lemma 26 we have for any  $\mathbf{a}_i \in \xi$

$$\mathcal{P}^n(x, A) \geq \mathbf{1}\{x \in \mathbf{a}_i\} \sum_{\mathbf{a}_j \in \xi_{\text{irr}}} Q_{i,j}^n \text{vol}(A | \check{\mathbf{a}}_j)$$

for all  $x \in M$ ,  $A \in \mathbb{B}$  and all  $n \geq 1$ . Therefore,

$$\begin{aligned} \overline{\mathcal{P}}(x, A) &= \frac{1}{N_\xi} \sum_{n=1}^{N_\xi} \mathcal{P}^n(x, A) \geq \mathbf{1}\{x \in \mathbf{a}_k\} \frac{1}{N_\xi} \sum_{n=1}^{N_\xi} \sum_{\mathbf{a}_j \in \xi_{\text{irr}}} Q_{k,j}^n \text{vol}(A | \check{\mathbf{a}}_j) \\ &\geq \mathbf{1}\{x \in \mathbf{a}_k\} \min_{\mathbf{a}_i \in \xi} \frac{1}{N_\xi} \sum_{n=1}^{N_\xi} \sum_{\mathbf{a}_j \in \xi_{\text{irr}}} Q_{i,j}^n \text{vol}(A | \check{\mathbf{a}}_j) \end{aligned}$$

for all  $\mathbf{a}_k \in \xi$  and all  $x$ . And since  $x$  is contained in at least one element of  $\xi$  we obtain the bound

$$\overline{\mathcal{P}}(x, A) \geq \min_{\mathbf{a}_i \in \xi} \frac{1}{N_\xi} \sum_{n=1}^{N_\xi} \sum_{\mathbf{a}_j \in \xi_{\text{irr}}} Q_{i,j}^n \text{vol}(A | \check{\mathbf{a}}_j)$$

uniformly in  $x \in M$  and  $A \in \mathbb{B}$ .

Now define the measure  $\psi$  on  $M$  by

$$\psi(A) = \min_{\mathbf{a}_i \in \xi} \frac{1}{N_\xi} \sum_{n=1}^{N_\xi} \sum_{\mathbf{a}_j \in \xi_{\text{irr}}} Q_{i,j}^n \text{vol}(A | \check{\mathbf{a}}_j) .$$

The choice  $N_\xi$  implies that

$$\psi(\check{\mathbf{a}}_k) = \min_{\mathbf{a}_i \in \xi} \frac{1}{N_\xi} \sum_{n=1}^{N_\xi} \sum_{\mathbf{a}_j \in \xi_{\text{irr}}} Q_{i,j}^n \text{vol}(A | \check{\mathbf{a}}_j) = \min_{\mathbf{a}_i \in \xi} \frac{1}{N_\xi} \sum_{n=1}^{N_\xi} Q_{i,k}^n \geq \min_{\mathbf{a}_i \in \xi} \frac{1}{N_\xi} Q_{i,k}^{N(\mathbf{a}_i \rightarrow \mathbf{a}_k)} > 0$$

for any  $\mathbf{a}_k \in \xi_{\text{irr}}$ . In particular, the measure  $\psi$  is non-trivial. Therefore,

$$\beta \varphi(A) = \psi(A) , \quad 1 \geq \beta = \psi(M) = \min_{\mathbf{a}_i \in \xi} \frac{1}{N_\xi} \sum_{n=1}^{N_\xi} \sum_{\mathbf{a}_j \in \xi_{\text{irr}}} Q_{i,j}^n > 0 ,$$

which finishes the proof.  $\square$

**Proposition 29** (Invariant measure for  $\mathcal{P}$  and  $\overline{\mathcal{P}}$ ; rate of convergence).

1. The Markov kernel  $\overline{\mathcal{P}}$  has a unique invariant probability measure  $\pi$ .

2. For any initial measure  $\mu_0$  on  $M$  the estimate

$$|\mu_0 \bar{\mathcal{P}}^n - \pi|_{\text{TV}} \leq (1 - \beta)^n$$

holds for all  $n \geq 1$ , where  $\beta$  is as in Lemma 28, and the total variation norm of a signed measure  $\nu$  is defined to be  $|\nu|_{\text{TV}} = \sup_{|A| \leq 1} \nu(A)$ .

3. The Markov kernel  $\mathcal{P}$  has a unique invariant probability measure, which is also given by  $\pi$ .

*Proof.* The first two claims are immediate consequences of the Doeblin condition for  $\bar{\mathcal{P}}$  proved in Lemma 28.

If  $\mu$  is an invariant probability measure for  $\mathcal{P}$ , then it clearly must be invariant for  $\bar{\mathcal{P}}$ . Therefore the first of the three claimed statements implies that  $\mathcal{P}$  can have at most one invariant measure, which must be  $\pi$ .

By invariance of  $\pi$  for  $\bar{\mathcal{P}}$  and  $\mathcal{P}\bar{\mathcal{P}} = \bar{\mathcal{P}}\mathcal{P}$  the identity  $\pi\mathcal{P} = \pi\bar{\mathcal{P}}^n\mathcal{P} = \pi\mathcal{P}\bar{\mathcal{P}}^n$  holds for all  $n \geq 1$ , so that the second of the claimed expressions shows that  $\pi\mathcal{P} = \lim_{n \rightarrow \infty} \pi\mathcal{P}\bar{\mathcal{P}}^n = \pi$ , which finishes the proof.  $\square$

### 4.3 Approximation of the stationary distribution

In what follows we assume that the perturbed system has a unique ergodic measure and that its support is strictly contained in  $M$ . Moreover, we assume that  $\mathcal{P}$  has a spectral gap  $0 < \theta \leq 1$  in the following sense. Let  $N \geq 1$  be fixed, and denote by

$$\bar{\mathcal{P}} = \sum_{k=1}^N \frac{1}{N} \mathcal{P}^k \tag{4a}$$

the Markov kernel corresponding to the sampled chain with uniform sampling distribution on  $\{1, \dots, N\}$ . The spectral gap property that we assume is that for any two probability measures  $\nu$  and  $\nu'$

$$|\nu \bar{\mathcal{P}}^n - \nu' \bar{\mathcal{P}}^n|_{\text{TV}} \leq C(1 - \theta)^n |\nu - \nu'|_{\text{TV}} \tag{4b}$$

for all  $n \geq 1$ , where  $C$  is some constant that does not depend on the choice of the measures  $\nu$  and  $\nu'$ .

**Lemma 30** (Sub-Markovian approximation). *Let  $\mathcal{Q}$  be a sub-Markov kernel on  $M$  such that  $\mathcal{Q} \leq \mathcal{P}$ , and introduce*

$$\kappa_- = \inf_{x \in M} [\mathcal{P}(x, M) - \mathcal{Q}(x, M)], \quad \kappa_+ = \sup_{x \in M} [\mathcal{P}(x, M) - \mathcal{Q}(x, M)]$$

which thus satisfy  $0 \leq \kappa_- \leq \kappa_+ \leq 1$ . Let  $\psi$  be a probability measure on  $M$ , and let  $\lambda \in \mathbb{R}$  be such that  $\lambda \psi = \psi \mathcal{Q}$ . Then the estimates

$$0 \leq \kappa_- \leq 1 - \lambda \leq \kappa_+ \leq 1 \quad \text{and} \quad |\pi - \psi|_{\text{TV}} \leq \frac{C}{\theta} \left[ 1 - \sum_{k=1}^N \frac{1}{N} (1 - \kappa_+)^k \right]$$

hold.

*Proof.* Since  $\pi$  is stationary for  $\mathcal{P}$ , it is also stationary for  $\bar{\mathcal{P}}$ . Therefore, we have that  $(\pi - \psi) \bar{\mathcal{P}} - (\pi - \psi) = \psi - \psi \bar{\mathcal{P}}$ , and hence

$$(\pi - \psi) \bar{\mathcal{P}}^n - (\pi - \psi) = \sum_{k=0}^{n-1} (\psi - \psi \bar{\mathcal{P}}) \bar{\mathcal{P}}^k$$

for any  $n \geq 1$ . Since  $\psi$  and  $\psi \bar{\mathcal{P}}$  are probability measures on  $M$ , the assumed spectral gap implies

$$|(\pi - \psi) \bar{\mathcal{P}}^n - (\pi - \psi)|_{\text{TV}} \leq \sum_{k=0}^{n-1} C (1 - \theta)^k |\psi - \psi \bar{\mathcal{P}}|_{\text{TV}} \leq \frac{C}{\theta} |\psi - \psi \bar{\mathcal{P}}|_{\text{TV}}$$

for all  $n \geq 1$ , and hence  $|\pi - \psi|_{\text{TV}} \leq \frac{C}{\theta} |\psi - \psi \bar{\mathcal{P}}|_{\text{TV}}$  by passing to the limit  $n \rightarrow \infty$ .

Furthermore, since  $\mathcal{Q}$  is sub-Markovian and  $\mathcal{Q} \leq \mathcal{P}$  we have that

$$\begin{aligned} \lambda &= \lambda \psi(M) = [\psi \mathcal{Q}](M) = [\psi \mathcal{P}](M) - \left[ [\psi \mathcal{P}](M) - [\psi \mathcal{Q}](M) \right] \\ &= 1 - \int \psi(dx) \left[ \mathcal{P}(x, M) - \mathcal{Q}(x, M) \right] \end{aligned}$$

and hence

$$0 \leq 1 - \kappa_+ \leq \lambda \leq 1 - \kappa_- \leq 1$$

follow for the upper and lower bounds on  $\lambda$ .

Finally, note that with

$$\bar{\mathcal{Q}} = \sum_{k=1}^N \bar{\mathcal{Q}}^k, \quad \psi \bar{\mathcal{Q}} = \bar{\lambda} \psi, \quad \bar{\lambda} = \sum_{k=1}^N \frac{1}{N} \lambda^k$$

it follows that

$$\psi \bar{\mathcal{P}} - \psi = \psi \bar{\mathcal{P}} - \psi \bar{\mathcal{Q}} + \psi \bar{\mathcal{Q}} - \psi = (\psi \bar{\mathcal{P}} - \psi \bar{\mathcal{Q}}) - (1 - \bar{\lambda}) \psi$$

where  $\psi \bar{\mathcal{P}} - \psi \bar{\mathcal{Q}}$  and  $(1 - \bar{\lambda}) \psi$  are positive measure of equal total mass  $1 - \bar{\lambda}$ . And since  $\bar{\mathcal{P}}$  is a Markov operator the trivial bound  $|\psi - \psi \bar{\mathcal{P}}|_{\text{TV}} \leq 1 - \bar{\lambda}$  given by the total mass implies

$$\begin{aligned} |\pi - \psi|_{\text{TV}} &\leq \frac{C}{\theta} |\psi - \psi \bar{\mathcal{P}}|_{\text{TV}} \leq \frac{C}{\theta} (1 - \bar{\lambda}) = \frac{C}{\theta} \left[ 1 - \sum_{k=1}^N \frac{1}{N} \lambda^k \right] \\ &\leq \frac{C}{\theta} \left[ 1 - \sum_{k=1}^N \frac{1}{N} (1 - \kappa_+)^k \right] \end{aligned}$$

and finishes the proof.  $\square$

#### 4.4 Time complexity of computing the ergodic measures

For sake of simplicity, from now on we assume  $M$  to be the  $d$ -dimensional cube  $[0, 1]^d$  and  $\zeta_\delta = \{\mathbf{a}_1, \dots, \mathbf{a}_{|\zeta|}\}$  to be a regular partition of diameter  $\delta$ . Because of regularity, all the atoms have the same volume  $\text{vol}(\mathbf{a}) = \delta^d$ . The volume of any  $\varepsilon$ -ball will be denote by  $\text{vol}(B_\varepsilon)$ .

Let  $\zeta$  be a partition of diameter  $\delta$ . We now describe how to construct a sub-Markov kernel  $P_\zeta^\varepsilon$  with a prescribed total mass deficiency.  $P_\zeta^\varepsilon$  will consist of a  $|\zeta| \times |\zeta|$  matrix whose entries will be either 0 or  $p = \frac{\text{vol} \mathbf{a}}{\text{vol} B_\varepsilon}$ . If the map  $f$  is poly-time computable, then each entry can be decided in polynomial time.

Let

$$m(\delta) := \sup\{d(f(x), f(y)) : x, y \in M, d(x, y) \leq \delta\} \quad (5)$$

be the uniform modulus of continuity of  $f$ . Then of course we have that

$$m(\delta) \searrow 0 \quad \text{as} \quad \delta \rightarrow 0$$

and

$$d(f(x), f(y)) \leq m(\delta) \quad \text{whenever} \quad d(x, y) \leq \delta.$$

**Proposition 31.**

$$\sup_{x \in M} [P(x, M) - P_\zeta^\varepsilon(x, M)] \leq C_M \frac{m(\delta) + 2\delta + 2\varepsilon}{\varepsilon}$$

where  $C_M$  is a constant which depends only on the manifold  $M$ .

*Proof.* Let  $x \in \mathbf{a}$ . Denote the density of  $P_\zeta^\epsilon(x, M)$  by  $\hat{p}_x(y)$ .

$$\begin{aligned}
P(x, M) - P_\zeta^\epsilon(x, M) &= \sum_{\mathbf{a}' \in \zeta} \int_{\mathbf{a}'} dy [p_x(y) - \hat{p}_x(y)] \\
&= \sum_{\mathbf{a}' \in \zeta} \int_{\mathbf{a}'} \frac{dy}{\text{vol}(B_\epsilon)} [\mathbf{1}\{y \in f(x)^\epsilon \cap \mathbf{a}'\} - \hat{p}_x(y)] \\
&= \sum_j \frac{1}{\text{vol}(B_\epsilon)} [\text{vol}(f(x)^\epsilon \cap \mathbf{a}_j) - \text{vol}(\mathbf{a}_j) \mathbf{1}\{d_\epsilon(c_j, \mathcal{S}_\epsilon(c_i)) < \epsilon - m(\delta) - \delta - 2\epsilon\}] \\
&\leq \sum_j \frac{\text{vol}(\mathbf{a}_j)}{\text{vol}(B_\epsilon)} [\mathbf{1}\{A\} - \mathbf{1}\{B\}] \quad (\text{where } A = \{d(c_j, \mathcal{S}_\epsilon(c_i)) < \epsilon + m(\delta) + \delta + \epsilon\} \\
&\text{and } B = \{d(c_j, \mathcal{S}_\epsilon(c_i)) < \epsilon - m(\delta) - \delta - 3\epsilon\}) \\
&= \sum_j \frac{\text{vol}(\mathbf{a}_j)}{\text{vol}(B_\epsilon)} \mathbf{1}\{\epsilon - m(\delta) - \delta - 3\epsilon \leq d(c_j, \mathcal{S}_\epsilon(c_i)) < \epsilon + m(\delta) + \delta + \epsilon\} \\
&\leq \frac{\text{vol}(B_{\epsilon+m(\delta)+2\delta+\epsilon}) - \text{vol}(B_{\epsilon-m(\delta)-3\epsilon-2\delta})}{\text{vol}(B_\epsilon)} \\
&\leq C_M \frac{m(\delta) + 2\delta + 2\epsilon}{\epsilon}.
\end{aligned}$$

□

## 5 Proof of Theorem C

### 5.1 Outline of the Proof

In the proof of Theorem B, we approximated the transfer operator by a finite matrix  $\{p_{i,j}\}$ , which corresponded more or less to the projection of the operator  $P$  on a finite partition  $\zeta$ . In this sense, this discretization was a “piece-wise constant” approximation of the operator  $P$ . In order to increase the precision of this approximation, and hence the precision  $\alpha = 2^{-n}$  of the computation of the invariant measure, we are forced to increase the resolution of the partition  $\zeta$ . This makes the size of the finite matrix approximation of  $P$  grow exponentially in  $n$ .

The idea in getting rid of this exponential growth, is to use a fixed partition  $\zeta$ , which will depend only on the noise  $K_\epsilon$ , and not on the precision  $n$ . Instead of using a “piece-wise constant” approximation, we represent the operator  $P$  *exactly* on each  $\mathbf{a} \in \zeta$  by a Taylor series. The regularity of the transition kernel implies the corresponding regularity of the push-forward of

any density. More precisely, if  $\rho^{(t)}$  denotes the density at time  $t$ , then

$$\begin{aligned}\rho^{(t)}(x) &= \sum_{\mathbf{a} \in \zeta} \mathbf{1}\{x \in \mathbf{a}\} \sum_{k=0}^{\infty} \rho_{\mathbf{a},k}^{(t)} (x - x_{\mathbf{a}})^k \\ \rho^{(t+1)}(x) &= \sum_{\mathbf{a}_i \in \zeta} \mathbf{1}\{x \in \mathbf{a}_i\} \sum_{l=0}^{\infty} \rho_{\mathbf{a}_i,l}^{(t+1)} (x - x_{\mathbf{a}_i})^l \\ \rho_{\mathbf{a}_i,l}^{(t+1)} &= \sum_{\mathbf{a}_j, m} \rho_{\mathbf{a}_j,m}^{(t)} \int_{\mathbf{a}_j} (y - x_{\mathbf{a}_j})^m \frac{\partial_2^l K_f(y, x_{\mathbf{a}_i})}{l!} dy .\end{aligned}$$

provides an infinite matrix representation of the transition operator in terms of its action on the Taylor coefficients of the densities. See Section 5.2.

The assumed analytic properties of the transition kernel allow us to truncate the power series representation of the densities (see Lemma 38), and represent the corresponding truncation  $P_N$  of the transition operator as a finite matrix.

We then show that the size of this matrix depends *linearly* on the bit-size  $n$  of the precision of the calculation of the invariant density (see Theorem 36 and Proposition 39). This is where the analytic properties of the kernel  $K_\varepsilon$  are used. The actual algorithm iterates  $P_N^{(t)}\rho$  of some initial density  $\rho$  sufficiently many times (linear in the bit size precision), and then uses the resulting vector to compute  $n$  significant bits of the the invariant density  $\pi(x)$  at some point  $x$  by using the Taylor formula

$$\sum_{k=1}^N P_N \rho^{(t)}(k) (x - x_{\mathbf{a}})^k .$$

This shows that the invariant density is an analytic poly-time computable function, and Proposition 11 finishes the proof.

We now give the technical details. As mentioned in the introduction, we consider only the one dimensional case.

## 5.2 A priori bounds

The standing assumptions on  $K_\varepsilon(y, x)$  in this section are:

**Assumption 32** (Uniform regularity of the transition kernel).

(i) There exists constants

$C > 0$  and  $\gamma > 0$  such that  $|\partial_2^k K_\varepsilon(y, x)| \leq C k! e^{\gamma k}$  for all  $k \in \mathbb{N}$  and all  $x, y \in M$ .

(ii)  $K_\varepsilon(f(\cdot), x)$  is poly-time integrable.

Since  $\varepsilon$  will be fixed, we will denote the kernel  $K_\varepsilon(f(y), x)$  just by  $K_f(y, x)$  to shorten the notation.

Let  $\mu$  be a probability measure on  $M$ . Recall that the transition operator is given by

$$\mu P(dx) = dx \rho(x), \quad \rho(x) = \int_M \mu(dy) K_\varepsilon(f(y), x), \quad (6)$$

and shows that  $\mu P(dx)$  has a density for any probability measure  $\mu$ .

**Lemma 33** (A priori regularity of  $\rho$ ).

- (i) The estimate  $\sup_{x \in M} |\partial^k \rho(x)| \leq C k! e^{\gamma k}$  holds for all  $k \in \mathbb{N}$ .
- (ii) For any partition  $\zeta$  satisfying  $e^\gamma \text{diam } \zeta < 1$  the density  $\rho$  admits for all  $x$  the series representation

$$\rho(x) = \sum_{\mathfrak{a} \in \zeta} \mathbf{1}\{x \in \mathfrak{a}\} \sum_{k=0}^{\infty} \rho_{\mathfrak{a},k} (x - x_{\mathfrak{a}})^k \quad \text{where} \quad |\rho_{\mathfrak{a},k}| \leq C e^{\gamma k},$$

which converges absolutely and exponentially fast, uniformly in  $x$ .

*Proof.* By definition of  $\rho(x)$  we have  $\partial^k \rho(x) = \int_M \mu(dy) \partial_2^k K_\varepsilon(f(y), x)$  for all  $k \in \mathbb{N}$  and all  $x \in M$ . Therefore, the claimed estimate on  $\sup_{x \in M} \partial^k \rho(x)$  follows from Assumption 32. Using this result the second claim follows from Taylor's theorem.  $\square$

Our method will further rely on the following assumption:

**Assumption 34** (Mixing assumption).

- (iv) There exists constants  $C > 0$  and  $\theta < 1$  such that

$$\left\| \frac{\mu P^t(dx)}{dx} - \frac{\nu P^t(dx)}{dx} \right\|_{\infty} \leq C \theta^t |\mu - \nu|_{\text{TV}} \leq 2C \theta^t \quad \text{for all } t \geq 1$$

holds for any two probability measures  $\mu$  and  $\nu$ .

Under Assumption 34 the Markov chain generated by  $P$  has a unique invariant measure, which we denote by  $\pi(dx)$ . Furthermore, it also follows that this measure has a bounded density with respect to the volume measure on  $M$ . By slightly abusing notation we will denote the density of the stationary measure by  $\pi(x)$ .

We now show the two facts above follow from assumption (i).

**Lemma 35** (Examples for  $K_\varepsilon$ ). *Part (i) of Assumption 32 is automatically satisfied, if the kernel  $K_\varepsilon(y, \cdot)$  is analytic, uniformly in  $y$ . If in addition there exist constants  $0 < c_- \leq c_+$  such that  $c_- \leq K_\varepsilon(y, x) \leq c_+$ , then Assumption 34 is satisfied.*

*Proof.* If  $K_\varepsilon(y, \cdot)$  is analytic, then  $K_\varepsilon(y, \cdot)$  admits an everywhere converging power series representation, which by compactness of  $M$  implies that there exist  $C(y) > 0$  and  $\gamma(y) > 0$  such that  $\sup_{x \in M} |\partial_2^k K_\varepsilon(y, x)| \leq C(y) k! e^{\gamma(y)k}$  for all  $k \in \mathbb{N}$ . The assumed uniformity of the analyticity simply means that  $C(y)$  and  $\gamma(y)$  can be uniformly chosen with respect to  $y$ , which proves the first part.

Now assume the existence of  $c_\pm$  as stated in the second part. Let  $\mu$  and  $\nu$  be two probability measures on  $M$ . From the definition of the transition operator (6)

$$\begin{aligned} \int_M [\mu P(dx) - \nu P(dx)] A(x) &= \int_M dx \int_M [\mu(dy) - \nu(dy)] K_f(y, x) A(x) \\ &= \int_M dx \int_M [\mu(dy) - \nu(dy)] [K_f(y, x) - c_-] A(x) \\ &= \theta \int_M dx \int_M [\mu(dy) - \nu(dy)] \frac{K_f(y, x) - c_-}{\theta} A(x), \quad \theta = 1 - |M| c_- < 1 \end{aligned}$$

for any bounded function  $A: M \rightarrow \mathbb{R}$ . The assumed lower bound implies that  $\frac{K_f(y, x) - c_-}{\theta}$  is a probability density (with respect to  $x$ ), and hence

$$|\mu P - \nu P|_{\text{TV}} \leq \theta |\mu - \nu|_{\text{TV}}$$

follows. Iterating this inequality we obtain

$$|\mu P^t - \nu P^t|_{\text{TV}} \leq \theta^t |\mu - \nu|_{\text{TV}} \leq 2\theta^t$$

for all  $t \geq 1$  and any two probability measures  $\mu$  and  $\nu$ . From the upper bound on the kernel it follows

$$\left\| \frac{\mu P(dx)}{dx} - \frac{\nu P(dx)}{dx} \right\|_\infty = \sup_{x \in M} \left| \int_M [\mu(dy) - \nu(dy)] K_f(y, x) \right| \leq c_+ |\mu - \nu|_{\text{TV}}$$

and hence

$$\left\| \frac{\mu P^t(dx)}{dx} - \frac{\nu P^t(dx)}{dx} \right\|_\infty \leq c_+ |\mu P^{t-1} - \nu P^{t-1}|_{\text{TV}} \leq c_+ \theta^{t-1} |\mu - \nu|_{\text{TV}}$$

as was to be shown.  $\square$

Because of Lemma 33 we can consider only densities satisfying the a priori bound, and we will do so. The density of at time  $t$  of a probability measure will be denoted by  $\rho^{(t)}(x)$ .

Using Lemma 33 we know that for any time  $t$ , such a density can be written as

$$\rho^{(t)}(x) = \sum_{\mathbf{a} \in \zeta} \mathbf{1}\{x \in \mathbf{a}\} \sum_{k=0}^{\infty} \rho_{\mathbf{a},k}^{(t)} (x - x_{\mathbf{a}})^k$$

and therefore

$$\begin{aligned} \rho^{(t+1)}(x) &= P\rho^{(t)}(x) = \int_M \rho(y) K_f(y, x) dy \\ &= \sum_{\mathbf{a}_j, m} \rho_{\mathbf{a}_j, m}^{(t)} \int_{\mathbf{a}_j} (y - x_{\mathbf{a}_j})^m K_f(y, x) dy . \end{aligned}$$

Expanding  $K_f$  gives

$$\begin{aligned} \rho^{(t+1)}(x) &= \sum_{\mathbf{a}_i \in \zeta} \mathbf{1}\{x \in \mathbf{a}_i\} \sum_{l=0}^{\infty} \rho_{\mathbf{a}_i, l}^{(t+1)} (x - x_{\mathbf{a}_i})^l \\ \rho_{\mathbf{a}_i, l}^{(t+1)} &= \sum_{\mathbf{a}_j, m} \rho_{\mathbf{a}_j, m}^{(t)} \int_{\mathbf{a}_j} (y - x_{\mathbf{a}_j})^m \frac{\partial_2^l K_f(y, x_{\mathbf{a}_i})}{l!} dy . \end{aligned}$$

We can therefore represent the operator  $P$ , acting on densities satisfying the a priori regularity, exactly by a matrix of size  $|\zeta| \times |\zeta|$ , whose entry  $P^{(\mathbf{a}_i, \mathbf{a}_j)}$  is in turn an infinite matrix with matrix elements

$$P^{(\mathbf{a}_i, \mathbf{a}_j)}(l, m) = \int_{\mathbf{a}_j} (y - x_{\mathbf{a}_j})^m \frac{\partial_2^l K_f(y, x_{\mathbf{a}_i})}{l!} dy , \quad l, m \geq 0 . \quad (7)$$

### 5.3 Truncation step

The idea here is to truncate the operator  $P$ , represented by the infinite matrix (7), by dropping the higher order terms. Recall Lemma 33 and corresponding representation of densities

$$\rho(x) = \sum_{\mathbf{a} \in \zeta} \mathbf{1}\{x \in \mathbf{a}\} \sum_{k=0}^{\infty} \rho_{\mathbf{a},k} (x - x_{\mathbf{a}})^k ,$$

with  $|\rho_{\mathbf{a},k}| \leq C e^{\gamma k}$  for all  $\mathbf{a}, k$ , where  $e^{\gamma} \text{diam } \zeta < 1$ . For any  $N \geq 1$  we define the **truncation projection**

$$\Pi_N \rho(x) := \sum_{\mathbf{a} \in \zeta} \sum_{k=0}^N \rho_{\mathbf{a},k} (x - x_{\mathbf{a}})^k , \quad \hat{\rho}_N(x) = \rho(x) - \Pi_N \rho(x) , \quad (8a)$$

where  $\hat{\rho}_N$  denotes the remainder term. Correspondingly, we define the truncated transition operator by

$$P_N := \Pi_N P \Pi_N, \quad (8b)$$

whose matrix elements are given by (7), with  $l, m = 1, \dots, N$ . A schematic representation of one application of the operator  $P_N$  is shown in Fig. 1.

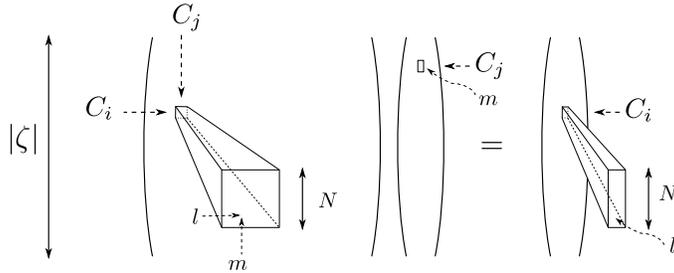


Figure 1: Graphical representation of the equation  $P_N \rho_N^{(t)} = \rho_N^{(t+1)}$ .

The following theorem states the desired linear dependence of both the number of iterations  $t$  and the number of Taylor coefficients  $N$  in the precision parameter  $n$ .

**Theorem 36.** *There exist linear functions  $t(n)$  and  $N(n)$  such that*

$$\| \pi - P_N^t \rho \|_\infty \leq 2^{-n}$$

for all  $n \in \mathbb{N}$ , uniformly in  $\rho$ .

*Proof.* We will need the following lemmas:

Let  $\mu$  be a probability measure with a density of the type of Lemma 33. Denote the densities of  $\mu P^t$  by  $\rho^{(t)}$  for all  $t \geq 0$ .

**Lemma 37.** *Then*

$$\Pi_N \rho^{(t)} - P_N^t \rho^{(0)} = \sum_{s=0}^{t-1} P_N^s Q_N \rho^{(t-1-s)}$$

holds, where  $Q_N := \Pi_N P - P_N = \Pi_N P (1 - \Pi_N)$ .

*Proof.* Observe that the identity  $\rho^{(t)} = P \rho^{(t-1)}$  can be rewritten as  $\Pi_N \rho^{(t)} = P_N \rho^{(t-1)} + Q_N \rho^{(t-1)}$ , so that  $\Pi_N \rho^{(t)} = P_N^t \rho^{(0)} + \sum_{s=0}^{t-1} P_N^s Q_N \rho^{(t-1-s)}$  follows by iteration.  $\square$

**Lemma 38** (Truncation bounds).

(i) For any bounded function  $\eta$  the estimate

$$\|\Pi_N P\eta\|_\infty \leq \left[1 + |M| C \frac{[e^\gamma \text{diam } \zeta]^{N+1}}{1 - e^\gamma \text{diam } \zeta}\right] \|\eta\|_\infty$$

holds for all  $N$ .

(ii) For any bounded function  $\eta$  the estimate

$$\|P_N^s \eta\|_\infty \leq \left[1 + |M| C \frac{[e^\gamma \text{diam } \zeta]^{N+1}}{1 - e^\gamma \text{diam } \zeta}\right]^s \|\Pi_N \eta\|_\infty$$

holds for all  $s \geq 0$  and all  $N$ .

*Proof.* By definition

$$\Pi_N P\eta(x) = \int_M dy \eta(y) \Pi_N^x K_f(y, x)$$

where the superscript  $x$  indicates that  $\Pi_N$  acts on the  $x$ -variable in  $K_f(y, x)$ . Therefore,

$$\begin{aligned} \|\Pi_N P\eta\|_\infty &\leq \|\eta\|_\infty \max_x \int dy |\Pi_N^x K_f(y, x)| \\ &\leq \|\eta\|_\infty \left[1 + \max_x \int dy |(1 - \Pi_N)^x K_f(y, x)|\right] \\ &\leq \left[1 + |M| C \frac{[e^\gamma \text{diam } \zeta]^{N+1}}{1 - e^\gamma \text{diam } \zeta}\right] \|\eta\|_\infty \end{aligned}$$

where used the normalization  $\int dy K_f(y, x) = 1$  of the kernel, and the a priori bound on the Taylor coefficients of  $K_f(y, x)$  with respect to  $x$ .

In particular, it follows that

$$\|P_N \eta\|_\infty = \|\Pi_N P \Pi_N \eta\|_\infty \leq \left[1 + |M| C \frac{[e^\gamma \text{diam } \zeta]^{N+1}}{1 - e^\gamma \text{diam } \zeta}\right] \|\Pi_N \eta\|_\infty$$

and therefore

$$\|P_N^s \eta\|_\infty \leq \left[1 + |M| C \frac{[e^\gamma \text{diam } \zeta]^{N+1}}{1 - e^\gamma \text{diam } \zeta}\right]^s \|\Pi_N \eta\|_\infty$$

for all  $s \geq 0$  by iteration, which finishes the proof.  $\square$

**Proposition 39.** *Let  $\rho$  be an arbitrary admissible density. For all  $N, t$*

$$\| \pi - P_N^t \rho \|_\infty \leq \left[ 1 + |M| q_N \right] e^{|M| q_N t} q_N t + q_N + 2 C \theta^t$$

where we set  $q_N = C \frac{[e^\gamma \text{diam } \zeta]^{N+1}}{1 - e^\gamma \text{diam } \zeta}$ .

*Proof.* Observe that for all  $t$  the identity  $P_N^t Q_N \rho = P_N^t P(\rho - \rho_N)$  holds by the definition of the  $P_N$  and  $Q_N$ , and therefore

$$\begin{aligned} \| P_N^s Q_N \rho \|_\infty &= \| P_N^s P(\rho - \rho_N) \|_\infty \\ &\leq \left[ 1 + |M| C \frac{[e^\gamma \text{diam } \zeta]^{N+1}}{1 - e^\gamma \text{diam } \zeta} \right]^s \| \Pi_N P(\rho - \rho_N) \|_\infty \\ &\leq \left[ 1 + |M| C \frac{[e^\gamma \text{diam } \zeta]^{N+1}}{1 - e^\gamma \text{diam } \zeta} \right]^{s+1} \| \rho - \rho_N \|_\infty \end{aligned}$$

holds for all  $s \geq 0$  and all  $N$ , by Lemma 38. Using the a priori bounds on the density  $\rho$  stated in Lemma 33 we obtain

$$\| P_N^s Q_N \rho \|_\infty \leq \left[ 1 + |M| C \frac{[e^\gamma \text{diam } \zeta]^{N+1}}{1 - e^\gamma \text{diam } \zeta} \right]^{s+1} C \frac{[e^\gamma \text{diam } \zeta]^{N+1}}{1 - e^\gamma \text{diam } \zeta}$$

for all admissible densities  $\rho$ , and all  $N$ .

Combining this uniform estimate with Lemma 37

$$\begin{aligned} \left\| \Pi_N \rho^{(t)} - P_N^t \rho^{(0)} \right\|_\infty &\leq \sum_{s=0}^{t-1} \left\| P_N^s Q_N \rho^{(t-1-s)} \right\|_\infty \\ &\leq \sum_{s=0}^{t-1} \left[ 1 + |M| C \frac{[e^\gamma \text{diam } \zeta]^{N+1}}{1 - e^\gamma \text{diam } \zeta} \right]^{s+1} C \frac{[e^\gamma \text{diam } \zeta]^{N+1}}{1 - e^\gamma \text{diam } \zeta} \\ &= \left[ \frac{1}{|M|} + C \frac{[e^\gamma \text{diam } \zeta]^{N+1}}{1 - e^\gamma \text{diam } \zeta} \right] \left( \left[ 1 + |M| C \frac{[e^\gamma \text{diam } \zeta]^{N+1}}{1 - e^\gamma \text{diam } \zeta} \right]^t - 1 \right) \end{aligned}$$

and therefore

$$\begin{aligned} \| \pi - P_N^t \rho \|_\infty &\leq \left\| \Pi_N \rho^{(t)} - P_N^t \rho \right\|_\infty + \left\| \rho^{(t)} - \Pi_N \rho^{(t)} \right\|_\infty + \left\| \pi - \rho^{(t)} \right\|_\infty \\ &\leq \left[ \frac{1}{|M|} + C \frac{[e^\gamma \text{diam } \zeta]^{N+1}}{1 - e^\gamma \text{diam } \zeta} \right] \left( \left[ 1 + |M| C \frac{[e^\gamma \text{diam } \zeta]^{N+1}}{1 - e^\gamma \text{diam } \zeta} \right]^t - 1 \right) \\ &\quad + C \frac{[e^\gamma \text{diam } \zeta]^{N+1}}{1 - e^\gamma \text{diam } \zeta} + 2 C \theta^t \end{aligned}$$

for all  $N, t$  and any admissible density  $\rho$ .

Finally, the inequality  $(1 + \xi)^t - 1 \leq e^{t\xi} t \xi$ , which holds for all  $\xi, t > 0$ , implies

$$\begin{aligned} \|\pi - P_N^t \rho\|_\infty &\leq \left[1 + |M| q_N\right] e^{|M| q_N t} q_N t + q_N + 2C \theta^t \\ q_N &= C \frac{[e^\gamma \text{diam } \zeta]^{N+1}}{1 - e^\gamma \text{diam } \zeta} \end{aligned}$$

which finishes the proof.  $\square$

Now we are in a position to finish the proof of Theorem 36. Fix  $k > 0$ . Note that the particular choices

$$t = \frac{k}{\log \frac{1}{\theta}}, \quad N + 1 \geq \frac{k + \log k}{\log \frac{1}{e^\gamma \text{diam } \zeta}} + \frac{0 \vee [\log(|M| C) - k]}{\log \frac{1}{e^\gamma \text{diam } \zeta}} + \frac{\log \frac{1}{1 - e^\gamma \text{diam } \zeta} - \log \log \frac{1}{\theta}}{\log \frac{1}{e^\gamma \text{diam } \zeta}}$$

combined with the estimate in Proposition 39 shows

$$\begin{aligned} \|\pi - P_N^t \rho\|_\infty &\leq \left[1 + |M| q_N\right] e^{|M| q_N t} q_N t + q_N + 2C \theta^t \\ &\leq \left[1 + |M| q_N t\right] e^{|M| q_N t} q_N t + q_N t + 2C \theta^t \\ &\leq C [3 + 2e] e^{-k} \leq 8.5 C e^{-k} \end{aligned}$$

so that setting  $k = n + \log[8.5 C]$  shows that these linear functions

$$\begin{aligned} t(n) &= \frac{1}{\log \frac{1}{\theta}} n + \frac{\log[8.5 C]}{\log \frac{1}{\theta}} \\ N(n) &= \frac{2}{\log \frac{1}{e^\gamma \text{diam } \zeta}} n + \frac{0 \vee [\log(|M| C) - \log(8.5 C) - n]}{\log \frac{1}{e^\gamma \text{diam } \zeta}} \\ &\quad + \frac{\log \frac{1}{1 - e^\gamma \text{diam } \zeta} - \log \log \frac{1}{\theta}}{\log \frac{1}{e^\gamma \text{diam } \zeta}} + \frac{2 \log[8.5 C]}{\log \frac{1}{e^\gamma \text{diam } \zeta}} - 1 \end{aligned}$$

will suffice for  $\|\pi - P_N^t \rho\|_\infty \leq 2^{-n}$  for all  $n$ .  $\square$

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