CODING DISCRETIZATIONS OF CONTINUOUS FUNCTIONS

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ABSTRACT. We consider several coding discretizations of continuous functions which reflect their variation at some given precision. We study certain statistical and combinatorial properties of the sequence of finite words obtained by coding a typical continuous function when the diameter of the discretization tends to zero. Our main result is that any finite word appears on a subsequence discretization with any desired limit frequency.

1. INTRODUCTION

Take a straight line in the plane and code it by a $0-1$ sequence as follows: each time it crosses an integer vertical line (that is, $x = n$ for some $n \in \mathbb{Z}$) write a 0 and each time it crosses an integer horizontal line ($y = n$ for some $n \in \mathbb{Z}$) write a 1. In the case of irrational slope the corresponding sequence is called a Sturmian sequence [A]. A classical result tells us that each word that appears in such a sequence has a limiting frequency. Moreover, the set of numbers occurring as limit frequencies can be completely described [B]. Recently similar codings have been considered for quadratic functions and limiting frequencies are calculated for words which appear [DTZ1] and [DTZ2].

In this article we ask the question if limiting frequencies can appear in more general circumstance: namely for typical, in the sense of Baire, continuous functions. For such functions it is not clear which kind of coding should be used. Here we propose three different notions of coding. For each of these codings we study two different questions: if all finite words can appear in a code or not, and if words in the code of a typical function can have a limiting frequency.

A *discretization system* of $[0,1]$ is a sequence $X_n := \{0 = x_1^n, x_2^n, ..., x_{N_n}^n = 1\} \subset [0,1]$ where,

(1) $X_1 \subset X_2 \subset ... \subset X_n \subset .... \subset [0,1]$,
(2) For each X_n, $x_i^n < x_{i+1}^n$ for all $1 \le i < N_n$,
(3) The *maximal resolution* $H_n := \max_{1 \le i < N_n} |x_{i+1}^n - x_i^n|$ converges to zero.

We denote by
$$h_n := \min_{1 \le i < N_n} |x_{i+1}^n - x_i^n|,$$
the *minimal resolution*. To each discretization system X_n, we associate the (uniform) discretization of the image space given by
$$Y_n := \{y_j^n = jh_n : j \in \mathbb{N}\}.$$

Let $f \in C([0,1])$. For each $x_i^n \in X_n$ there is a unique $j \in \mathbb{Z}$ such that $f(x_i^n) \in [y_j^n, y_{(j+1)}^n)$. Let us denote this j by f_i^n.

Now we introduce the three different codings. Our notation for finite words will be as follows. If v is a finite word over some alphabet, then $|v|$ denotes its length.

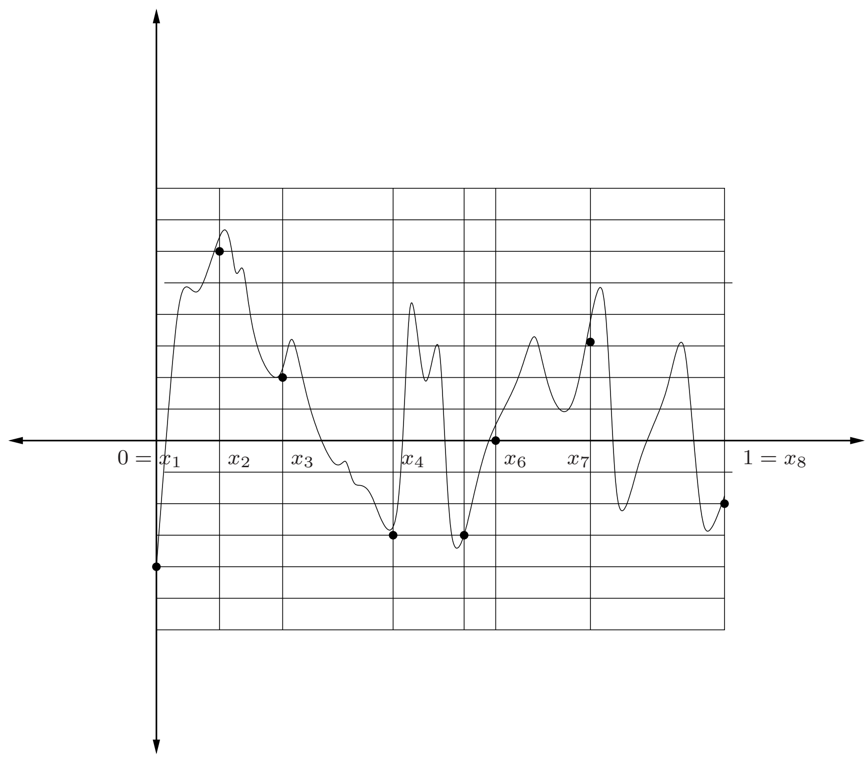

FIGURE 1. The various codes considered:
quantitative: $Q(f,n) = 10^{-}4^{-}5033^{-}5$
qualitative: $q(f,n) = 1^{-}1^{-}1011^{-}1$
stretched: $s(f,n) = 11111111110^{-}1^{-}1^{-}1^{-}10^{-}1^{-}1^{-}1^{-}1^{-}10011101110^{-}1^{-}1^{-}1^{-}1^{-}10$

For $0 \leq i \leq |v| - 1$, $v_{[i]}$ will denote the letter in position i. Thus, the expression $v_{[j \cdots k]}$ denotes the word $v_{[j]} v_{[j+1]} ... v_{[k]}$ (for $0 \leq j \leq k \leq |v| - 1$).

Definition 1. (See Figure 1) The *quantitative code* $Q(f,n) \in \mathbb{Z}^{N_n - 1}$ of $f \in C([0,1])$ is defined by:

$$Q(f,n)_{[i]} := f^n_{i+2} - f^n_{i+1}, \qquad 0 \leq i < N_n - 1.$$

The *qualitative* version $q(f,n) \in \{-1, 0, 1\}^{N_n - 1}$ of the quantitative code $Q(f,n)$ is defined by setting

$$q(f,n)_{[i]} := \begin{cases} 1, & \text{if } Q(f,n)_{[i]} > 0 \\ 0, & \text{if } Q(f,n)_{[i]} = 0 \\ -1, & \text{if } Q(f,n)_{[i]} < 0. \end{cases}$$

Finally, the *stretched* version $s(f,n) \in \{1, 0, -1\}^*$ of the quantitative code $Q(f,n)$ is defined as follows: if $Q(f,n)_{[i]}$ is positive then we replace it by a run of $Q(f,n)_{[i]}$ 1's followed by a zero and by a run of -1's followed by a zero if $Q(f,n)_{[i]}$ is negative.

All three of these codes seem natural in terms of discrete curves on the computer screen. In case when the discretization system is uniform, the stretched quantitative code of a line segment with irrational slope is exactly the well known coding by Sturmian sequences [A].

Let us introduce some more notation in order to state our main results. Let w, v be finite words over the same alphabet Σ (finite or infinite) such that $|w| \leq |v|$. We denote by

$$\mathit{oc}\,(w,v) := \#\{j : v_{[j \cdots j+|w|-1]} = w, 0 \leq j \leq |v| - |w|\} \tag{1}$$

the number of times w occurs in v and by

$$\mathit{fr}\,(w,v) := \frac{\mathit{oc}\,(w,v)}{|v|}$$

the relative frequency of w in v. The *minimal periodic factor length* $p(w)$ of w is defined to be

$$p(w) := \min\{|u| : \mathit{oc}\,(w, wu) = 2\}.$$

For example, $p(010) = 2$.

Remark 1. Let $\{v_n\}_{n\in\mathbb{N}}$ be an infinite sequence of finite words and let w be another finite word over the same alphabet such that $|w| \leq |v_n|$, for all $n \in \mathbb{N}$. Then, the limit relative frequency of w in $\{v_n\}_{n\in\mathbb{N}}$ is at most $\frac{1}{p(w)}$. That is,

$$\limsup_n \mathit{fr}\,(w, v_n) \leq \frac{1}{p(w)}.$$

A certain property on a complete metric space is said to be *typical* (or *topologically generic*) if the set on which it holds contains a G_δ dense set (that is, a countable intersection of open dense sets). Such sets are considered "large" from a topological point of view. The Baire category theorem (see [O]) ensures that a typical property holds on a dense subset of the space. For instance, it has long been known that the nowhere differentiable functions are topologically generic in $C[0,1]$. This result was proved originally by Banach [Ba] and Mazurkiewicz [M].

Our main result is the following:

Theorem 1. *Let X_n be a discretization system. For a typical $f \in C([0,1])$ the following holds:*

(i) *(Qualitative) For any $w \in \{-1,0,1\}^*$ and $\alpha \in [0, \frac{1}{p(w)}]$, there exists a subsequence n_i such that*

$$\lim_{i\to\infty} \mathit{fr}(w, q(f, n_i)) = \alpha.$$

(ii) *(Quantitative) Suppose that X_n satisfies $\liminf_n nh_n = 0$. Then for any $w \in \mathbb{Z}^*$ and $\alpha \in [0, \frac{1}{p(w)}]$, there exists a subsequence n_i such that*

$$\lim_{i\to\infty} \mathit{fr}(w, Q(f, n_i)) = \alpha.$$

(iii) *(Stretched) Suppose that X_n satisfies $\liminf_n nh_n = 0$ and $\frac{H_n}{h_n}$ is bounded. Then*

$$\liminf_{n\to\infty} \mathit{fr}(0, s(f,n)) = 0,$$

if $f(1) \geq f(0)$

$$\liminf_{n\to\infty} \boldsymbol{fr}(1, s(f,n)) = \limsup_{n\to\infty} \boldsymbol{fr}(-1, s(f,n)) = \frac{1}{2},$$

and if $f(1) \leq f(0)$

$$\limsup_{n\to\infty} \boldsymbol{fr}(1, s(f,n)) = \liminf_{n\to\infty} \boldsymbol{fr}(-1, s(f,n)) = \frac{1}{2}$$

2. Preliminaries

We start by a simple result, which says that one can focus on functions which do not intersect the discretization.

Lemma 1. *Let X_n be a discretization system. Then for a typical function f one has that for all $n \in \mathbb{N}$ and all $i = 1, ..., N_n$, $f(x_i^n) \in (y_j^n, y_{j+1}^n)$, for the corresponding $j \in \mathbb{N}$.*

Proof. The set $F_n = \{f : f(x_i^n) \neq y_j^n \text{ for all } j \in \mathbb{N} \text{ and } i = 1, ..., N_n\}$ is clearly open and dense. Hence, $\bigcap_n F_n$ is a G_δ-dense set. $\square$

One would expect that codings of typical functions contains few zeros and all possible words of 1's and $-1's$. This is partially true.

Proposition 1. *Let X_n be a discretization system. For a typical f, $q(f,n)$ contains no 0 for infinitely many n.*

Proof. We prove that the set of functions such that for all $n \in \mathbb{N}$, there exists $m \geq n$ such that $q(f,m)_{[i]} \neq 0$ for all $i = 0, ..., N_m - 2$, is residual in $C([0,1])$, i.e. it is the complement of a countable union of nowhere dense sets. Observe that $q(f,n)_{[i]} \neq 0$ whenever $|f(x_{i+1}^n) - f(x_i^n)| > h_n$. Clearly, the set

$$F^m = \{f : |f(x_{i+1}^m) - f(x_i^m)| > h_m \text{ for all } i = 1, ..., N_m - 1\}$$

is open. Moreover, for each $n \in \mathbb{N}$, the set

$$\bigcup_{m \geq n} F^m$$

is a dense open set. Indeed, given $g \in C([0,1])$ and $\varepsilon > 0$, there exists $m \geq n$ such that $h_m < \varepsilon$ and it is easy to construct a function $f \in F^m$ such that $\|g - f\|_\infty < \varepsilon$. Therefore,

$$\bigcap_n \bigcup_{m \geq n} F^m$$

is a G_δ-dense set.

$\square$

Remark 2. In the previous result, the symbol 0 cannot be replaced by 1 nor by -1. On the other hand, Theorem 1 says that the qualitative and the quantitative codings of a typical function is not statistical regular. So that from a statistical viewpoint, the symbols 1 or -1 (or any $n \in \mathbb{Z}$ in the quantitative case) are not privileged with respect to 0.

2.1. **Approximation by ε-boxes.** Here we will describe a simple construction which will be used in the proofs of our main results.

Let $\delta > 0$. For a given n we define a subdiscretization

$$X_n^\delta := \{x_{i_k} : k = 1, ..., K\}$$

of X_n as follows:

$$\begin{aligned} x_{i_1} &= 0, \\ x_{i_{k+1}} &= \max\{x_i^n \in X_n : x_i^n < x_{i_k} + 2\delta\}, \\ x_{i_K} &= 1. \end{aligned}$$

The number of points of X_n in the interval $(x_{i_k}, x_{i_{k+1}}]$ will be denoted by l_k. With this notation we have $x_{i_{k+1}} = x_{i_k + l_k}$.

Next, to each $g \in C([0,1])$ and $\varepsilon > 0$, the associated *ε-boxes* $B_k(g, \varepsilon, \delta)$ are defined by:

$$B_k(g, \varepsilon, \delta) := (x_{i_k}, x_{i_k} + 2\delta) \times (g(\Delta_k) - \frac{\varepsilon}{2}, g(\Delta_k) + \frac{\varepsilon}{2})) \tag{2}$$

where $\Delta_k = \frac{x_{i_k} + x_{i_{k+1}}}{2}$. See figure 2.1. We shall write just B_k when no confusion is possible.

Let $\delta_g : \mathbb{R}^+ \to \mathbb{R}^+$ denote the modulus of continuity of g. That is, for every x, x' in $[0,1]$, if $|x - x'| < \delta_g(\varepsilon)$ then $|f(x) - f(x')| < \varepsilon$.

Lemma 2. *For the ε-boxes $B_k(g, \varepsilon, \delta)$, $k = 1, ..., K$, the following holds:*

(i) *If $\delta < \delta_g(\frac{\varepsilon}{2})$ then*

$$(x, y) \in \bigcup_k B_k(g, \varepsilon, \delta_g(\frac{\varepsilon}{2})) \implies |g(x) - y| < \varepsilon.$$

That is, the ε-boxes form an ε-cover of the graph of g.

(ii) *If $\lceil \frac{1}{2\delta} \rceil H_n < 2\delta$, then $K = \lceil \frac{1}{2\delta} \rceil + 1$.*

Proof. (i). Let $(x, y) \in \bigcup_k B_k$. Then $(x, y) \in B_k$ for some k. We have then that

$$|x - \Delta_k| < \delta < \delta_g(\frac{\varepsilon}{2}),$$

which implies $|g(x) - g(\Delta_k)| < \frac{\varepsilon}{2}$. Since

$$g(\Delta_k) - \frac{\varepsilon}{2} < y < g(\Delta_k) + \frac{\varepsilon}{2}$$

we conclude $|g(x) - y| < \varepsilon$.

(ii). Since $x_{i_{k+1}} < x_{i_k} + 2\delta$, we have that $K \geq \lceil \frac{1}{2\delta} \rceil + 1$. Now, for each k we have $x_{i_k} + 2\delta - x_{i_{k+1}} \leq H_n$. It follows that

$$K \leq \left\lceil \frac{1}{2\delta} \right\rceil + \left\lceil \frac{\lceil \frac{1}{2\delta} \rceil H_n}{2\delta} \right\rceil.$$

Hence, if $2\delta > \lceil \frac{1}{2\delta} \rceil H_n$ we obtain $K \leq \lceil \frac{1}{2\delta} \rceil + 1$. $\square$

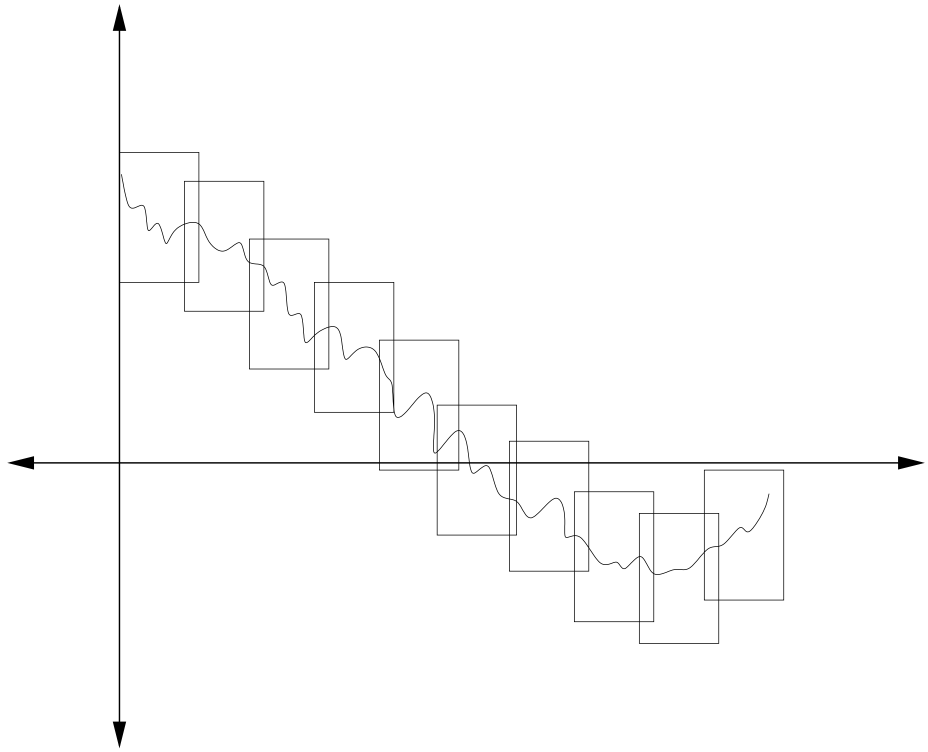

FIGURE 2. An ε-cover by the boxes $B_k(g, \varepsilon, \delta)$

2.2. **Words and frequencies.** Here we make a couple of simple observations that will be used in the next section.

Remark 3. Let Σ be an alphabet with more than two letters and consider a finite word w over Σ. For any integer $l \geq |w|$ and any i in the set

$$\left\{0, 1, ..., \left\lfloor \frac{l - |w|}{p(w)} \right\rfloor + 1\right\},$$

it is easy to construct a word v of length l such that $\boldsymbol{fr}(w, v) = i/l$. In fact, all the possible frequencies of w in words of length l are of this form. Therefore, for each $\alpha \in [0, \frac{1}{p(w)}]$ and $\epsilon > 0$, it is easy to construct a word v with prescribed length $|v| = l$ satisfying

$$|\boldsymbol{fr}(w, v) - \alpha| < \epsilon$$

provided that the length l is large enough.

Lemma 3. *Let w be a finite word, K be a natural number and $t > 0$. Let $\{v_k\}_{k=1}^{K}$ be a finite list of words such that $|\boldsymbol{fr}(w, v_k) - \alpha| < \frac{1}{3t}$. Let $a_1, a_2, ..., a_K$ be a list of K letters in Σ and consider the word v defined by*

$$v = v_1 a_1 v_2 a_2 \cdots v_K a_K.$$

Then, if $\frac{K|w|}{|v|} < \frac{1}{3t}$ it holds $|\boldsymbol{fr}(w, v) - \alpha| < \frac{1}{t}$.

Proof. Put $\boldsymbol{oc}(w, v_k) = p_k$ and $|v_k| = l_k$. Then we have

$$\frac{\sum_{k=1}^{K} p_k}{|v|} \leq \boldsymbol{fr}(w, v) \leq \frac{\sum_{k=1}^{K} p_k}{|v|} + \frac{K|w|}{|v|} \tag{3}$$

A simple calculation yields

$$\frac{\sum_{k=1}^{K} p_k}{|v|} = \frac{\sum_{k=1}^{K} p_k}{\sum_{k=1}^{K} l_k} - \frac{\sum_{k=1}^{K} p_k}{(\sum_{k=1}^{K} l_k)^2 + K\sum_{k=1}^{K} l_k}. \tag{4}$$

On the one hand we have:

$$\left|\frac{\sum_{k=1}^{K} p_k}{\sum_{k=1}^{K} l_k} - \alpha\right| = \left|\sum_{k=1}^{K} \boldsymbol{fr}(w, v_k)\frac{l_k}{l} - \alpha\right| < \frac{1}{3t}$$

and on the other hand, the absolute value of the second term in the right side of equation (4) is less than

$$\frac{|v| - K}{(|v| - K)^2 + K(|v| - K)} = \frac{1}{|v|} \leq \frac{1}{3t},$$

so that

$$\left|\frac{\sum_{k=1}^{K-1} p_k}{|v|} - \alpha\right| < \frac{2}{3t}.$$

Since $\frac{K|w|}{|v|} < \frac{1}{3t}$, from equation (3) we obtain $|\boldsymbol{fr}(w, v) - \alpha| < \frac{1}{t}$ and the lemma is proved. $\square$

3. Proofs

Proof of Theorem 1. We begin by proving parts (i) and (ii). For each finite word w in $\{-1, 0, 1\}^*$ or $\mathbb{Z}^*$, let $\{\alpha_s\}_{s\in\mathbb{N}}$ be a sequence which is dense on $[0, \frac{1}{p(w)}]$. Let F_i denote the open sets defined in Lemma 1 (the set of functions which do not intersect the discretization X_i). For integers s, n, t, consider the sets

$$\overline{F}^q_{w,s,n,t} := \{f \in \cap_{i\leq n} F_i : |\boldsymbol{fr}(w, q(f, n)) - \alpha_s| \leq \frac{1}{t}\},$$

$$\overline{F}^Q_{w,s,n,t} := \{f \in \cap_{i\leq n} F_i : |\boldsymbol{fr}(w, Q(f, n)) - \alpha_s| \leq \frac{1}{t}\}.$$

Clearly these sets are open since a function f in $\cap_{i\leq n} F_i$ can be perturbed without changing its code $q(f, n)$ or $Q(f, n)$. Hence, the following sets are open too.

$$F^q_{w,s,m,t} := \{f : \exists n \geq m, f \in \overline{F}^q_{w,s,n,t}\},$$

$$F^Q_{w,s,m,t} := \{f : \exists n \geq m, f \in \overline{F}^Q_{w,s,n,t}\}.$$

We now show that these sets are moreover dense. Let $g \in C([0, 1])$ and $\varepsilon > 0$. We will construct a function f in $F^q_{w,s,m,t}$ (respectively $F^Q_{w,s,m,t}$) such that $\|f - g\|_\infty \leq \varepsilon$.

Case $\boldsymbol{F^q_{w,s,m,t}}$. Put $\delta < \min\{\delta_g(\frac{\varepsilon}{2}), \frac{\varepsilon}{4}\}$ and let B_k be the associated ε-boxes. For $n \geq m$ large enough (in particular such that $\lceil\frac{1}{2\delta}\rceil H_n < 2\delta$) there exists a sequence of finite words v_k, $k = 1, ..., K - 1$, such that $|v_k| = l_k - 1$, $|\boldsymbol{fr}(w, v_k) - \alpha_s| < \frac{1}{3t}$ and $\frac{(K-1)|w|}{N_n} < \frac{1}{3t}$ (see Remark 3).

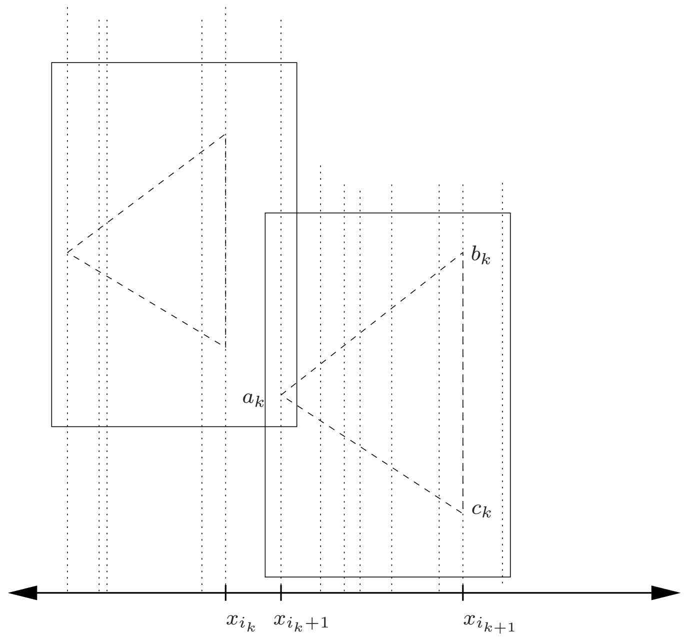

FIGURE 3. The triangles (a_k, b_k, c_k).

We claim that a function f_v can be constructed such that for each k we have $q(f_v, n)_{[i_k+1 \cdots i_k+l_k-1]} = v_k$ and f_v is ε-close to g (the interval (x_{i_k}, x_{i_k+1}) is reserved to make "the bridge" and there are K such intervals, see figure 3). To see this, observe that the condition $2\delta < \frac{\varepsilon}{2}$ implies that for each k, the triangles of vertices (a_k, b_k, c_k) defined by

$$
\begin{aligned}
a_k^q &= (x_{i_k+1}, g(\Delta_k)) \\
b_k^q &= (x_{i_{k+1}}, g(\Delta_k) + |v_k| h_n) \\
c_k^q &= (x_{i_{k+1}}, g(\Delta_k) - |v_k| h_n)
\end{aligned}
$$

are included in B_k and that for any v_k, a function f_v such that $q(f_v, n)_{[i_k+1 \cdots i_k+l_k]} = v_k$ can be inscribed in these triangles. By lemma 2 a function f so constructed satisfies $\|f - g\|_\infty \leq \varepsilon$. By lemma 3 we have that $|\boldsymbol{fr}(w, q(f, n)) - \alpha_s| < \frac{1}{t}$.

Case $\boldsymbol{F^Q_{w,s,m,t}}$. The proof that these sets are dense is the same as for the sets $F^q_{w,s,m,t}$, with the only exception that we have to take $\delta < \frac{\varepsilon}{4H(w)}$ where $H(w) := \max_i |w_i|$ denotes the *hight* of w. This condition assures that a function f_v such that $Q(f_v, n)_{[i_k+1 \cdots i_k+l_k-1]} = v_k$ can be inscribed in the corresponding triangles:

$$\begin{aligned} a_k^Q &= (x_{i_k+1}, g(\Delta_k)) \\ b_k^Q &= (x_{i_{k+1}}, g(\Delta_k) + |v_k| H(w) h_n) \\ c_k^Q &= (x_{i_{k+1}}, g(\Delta_k) - |v_k| H(w) h_n). \end{aligned}$$

It follows that the sets

$$\bigcap_{w,s,m,t} F^q_{w,s,m,t} \quad \text{and} \quad \bigcap_{w,s,m,t} F^Q_{w,s,m,t}$$

are both G_δ-dense.

Finally we prove part (iii). Let $u_n := \mathbf{oc}(1, s(f,n))$ be the numbers of 1's (or "ups") in $s(f,n)$ and $d_n := \mathbf{oc}(-1, s(f,n))$ be the number of -1's (or "downs"), in stage n. Then $V_n = u_n + d_n$ denotes the *total n-variation*. By definition we have that $|s(f,n)| = V_n + N_n$, where N_n is both the cardinality of the discretization and the number of zeros. Hence we have

$$\mathbf{fr}(1, s(f,n)) = \frac{u_n}{V_n + N_n}.$$

We will need the following lemma:

Lemma 4. *Let X_n be a discretization system satisfying* $\liminf_n nh_n = 0$. *Then, for a typical f, there are infinitely many n such that $u_n > \frac{nN_n}{3}$ and $d_n > \frac{nN_n}{3}$.*

Proof. Consider the set of functions

$$\overline{F}_n := \left\{ f : card\{i : Q(f,n)_i > n\} > \frac{N_n}{3} \text{ and } card\{i : Q(f,n)_i < -n\} > \frac{N_n}{3} \right\} \cap \bigcap_{i \le n} F_i$$

This is an open set. Moreover, for any $m \in \mathbb{N}$, the set

$$\bigcup_{n \ge m} \overline{F}_n$$

is dense. For let $g \in C[0,1]$ and consider the associated ε-boxes B_k. It is clear that for some $n \ge m$ such that $nh_n < \frac{\varepsilon}{2}$ one can construct a function f satisfying $graph(f) \subset \cup_k B_k$ and $|f(x_{i+1}) - f(x_i)| > n$ for all i. Moreover, we can alternate the sign of $|f(x_{i+1}) - f(x_i)|$ at every i, with at most K exceptions. Hence the function so constructed belongs to $\overline{F}_n$ and then the set

$$\bigcap_m \bigcup_{n \ge m} \overline{F}_n$$

is G_δ dense.

□

Denote by $\Delta = f(1) - f(0)$ the global growth of f. At resolution n, this quantity corresponds to $\Delta_n = u_n - d_n$. Since the vertical discretization is uniform of size h_n, we have that Δ_n equals either $\lfloor \frac{(f(1)-f(0))}{h_n} \rfloor$ or $\lfloor \frac{(f(1)-f(0))}{h_n} \rfloor + 1$ (depending on the position of the discretization). A simple calculation yields

$$\frac{u_n}{V_n} = \frac{1}{2 - \frac{\Delta_n}{u_n}}.$$

So, if $f(1) = f(0)$ we have $\frac{u_n}{V_n} = \frac{1}{2}$. Let M be a bound for $\frac{H_n}{h_n}$. We have then that $\frac{1}{h_n} \le MN_n$ and hence $\Delta_n \le (f(1) - f(0))MN_n + 1$. By Lemma 4 we have that

$$\frac{(f(1)-f(0))MN_n}{u_n} < \frac{3(f(1)-f(0))MN_n}{nN_n}$$

and

$$\frac{N_n}{V_n} < \frac{3}{n}$$

for infinitely many n, so that,

$$\liminf \frac{\Delta_n}{u_n} = 0 \quad \text{if} \quad f(1) > f(0), \tag{5}$$

$$\limsup \frac{\Delta_n}{u_n} = 0 \quad \text{if} \quad f(1) < f(0). \tag{6}$$

Hence, when $f(1) > f(0)$ we have

$$\liminf_{n\to\infty} \frac{u_n}{V_n + N_n} = \liminf_{n\to\infty} \frac{u_n}{V_n} = \frac{1}{2 - \liminf \frac{\Delta_n}{u_n}} = \frac{1}{2}$$

and when $f(1) < f(0)$ we have

$$\limsup_{n\to\infty} \frac{u_n}{V_n + N_n} = \limsup_{n\to\infty} \frac{u_n}{V_n} = \frac{1}{2 - \liminf \frac{\Delta_n}{u_n}} = \frac{1}{2}$$

and the results follows by symmetry.

$\square$

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