DYNAMICS AND ABSTRACT COMPUTABILITY: COMPUTING INVARIANT MEASURES.

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Abstract. We consider the question of computing invariant measures from an abstract point of view. Here, computing a measure means finding an algorithm which can output descriptions of the measure up to any precision. We work in a general framework (computable metric spaces) where this problem can be posed precisely. We will find invariant measures as fixed points of the transfer operator. In this case, a general result ensures the computability of isolated fixed points of a computable map. We give general conditions under which the transfer operator is computable on a suitable set. This implies the computability of many "regular enough" invariant measures and among them many physical measures.

On the other hand, not all computable dynamical systems have a computable invariant measure. We exhibit two examples of computable dynamics, one having a physical measure which is not computable and one for which no invariant measure is computable, showing some subtlety in this kind of problems.

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1. Introduction

An important fact motivating the study of the statistical properties of dynamical systems is that the pointwise long time prediction of a chaotic system is not possible, whereas, in many cases, the estimation or forecasting of averages and other long time statistical properties is. This often corresponds in mathematical terms to computing invariant measures, or estimating some of their properties.

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Giving a precise meaning to the computation of a continuous object like a measure is not a completely obvious task and involves the definition of effective versions of several concepts from mathematical analysis.

Our approach will be mainly based on the concept of computable metric space. To give a first example, let us consider the set $\mathbb{R}$ of real numbers. Beyond $\mathbb{Q}$ there are many other real numbers that can be handled by algorithms: π or $\sqrt{2}$ for instance can be approximated *at any given precision* (with rational numbers) by an algorithm. We can then identify a number with the algorithm that calculates it (if any) or, more precisely, to the string representing the program that approximates it. This set of points is called the set of *computable real numbers* and was introduced in the famous paper [35].

This kind of construction can be generalized to virtually any other separable metric space, taking a dense countable subset to play the same role as the rationals in the above example. Then, *computable* (or *recursive*, or *effective*) counterparts of most mathematical notions can be defined, and rigorous statements about the algorithmic approximation of abstract objects can be made, obtaining algorithmic versions of many classical theorems (see Section 2). In particular, this general approach gives the possibility to treat computation in measures spaces in a simple way, defining computable measures and computable functions between those spaces (and applying it to transfer operators). This will be the main theme of this paper.

The paper is devoted to the problem of computation of invariant measures in discrete time dynamical systems. By discrete time dynamical system we mean a system (X, T) were X is a metric space and $T : X \to X$ is a Borel measurable transformation. Here an invariant measure is a Borel probability measure μ on X such that for each measurable set A it holds $\mu(A) = \mu(T^{-1}(A))$. Such measures contain information on the statistical behavior of the system (X, T) and on the possible behavior of averages of observables along typical trajectories of the system. The map T moreover induces a function $L_T : PM(X) \to PM(X)$, where $PM(X)$ is the set of Borel probability measures over X endowed with a suitable metric (for details see Section 2.5). L_T is called the transfer operator associated to T (definition and basic results about this are reminded in Section 3).

Before entering into details about the computation of measures and invariant measures in particular, we remark that whatever we mean by "approximating a measure by an algorithm", there are only countably many "measure approximating algorithms" whereas, in general, a dynamical system may have uncountably many invariant measures (usually an infinite dimensional set). So, a priori most of them will not be algorithmically describable. This is not a problem because we should put our attention on the most "meaningful" ones. An important part of the theory of dynamical systems is indeed devoted to the understanding of "physically" relevant invariant measures. Informally speaking, these are measures which represent the asymptotic statistical behavior of "many" (positive Lebesgue measure) initial conditions, see Section 3. The existence and uniqueness of physical measures is a widely studied problem (see [37]), which has been solved for some important classes of dynamical systems. These measures are the good candidates to be computed.

Let us precise the concept of computable measure. As mentioned before, the framework of computable analysis can be applied to abstract metric spaces as the space $PM(X)$. A measure μ is then *computable* if it is a computable point of this measure space. In this case there is an algorithm such that, for each rational ε given as input, outputs a "finitely representable" measure (a finite rational convex combination of Dirac measures supported on "rational" points) which is ε-close to μ.

In the literature, there are several works dealing with the problem of approximating invariant measures, more or less informally from the algorithmic point of view (see e.g. [29, 25, 27, 31, 15, 16]). In these works the main technique consists in an adequate discretization of the problem. More precisely, in several of the above works the transfer operator associated to the dynamics is approximated by a finite dimensional one and the problem is reduced to the computation of the corresponding relevant eigenvectors (some effective convergence result then validates the quality of the approximation).

Another strategy to face the problem of computation of invariant measures consist in following the way the measure μ can be constructed and check that each step can be realized in an effective way. In some interesting examples we can obtain the physical measure as limit of iterates of the Lesbegue measure $\mu = \lim_{n\to\infty} L_T^n(m)$ where m is the Lesbegue measure and L_T is the transfer operator associated to T. To prove computability of μ the main point is to recursively estimate the speed of convergence to the limit. This sometimes can be done using the decay of correlations (see [20] where computability of physical measures in uniformly hyperbolic systems is proved in this way, see [22] for general relations between convergence of measures and decay of correlations with a point of view similar to the one of the present paper).

Let us illustrate the main results of the paper. The pair (X, T) is called a computable dynamical system, provided X is a computable metric space and $T : X \to X$ a computable transformation (for the precise definitions see Section 2.3). In this context, the general problem we are facing can be stated in the following terms:

Problem 1.

- **a)** *Given a computable dynamical system (X, T), does it admit computable invariant measures?*
- **b)** *Can they be found in an algorithmic way, starting from the description of the system?*

We will see that even the above question a) does not always have a positive answer. However, in many interesting situations, both of the above problems can be positively solved.

We will take an abstract point of view finding the interesting invariant measure as a fixed point of the transfer operator, giving general conditions ensuring its computability. The main tool for this purpose will be the following statement (we give it informally, see Theorems 3.1.1 and 3.2.1 for precise statements).

Theorem A *Let X be a computable metric space and T a function which is computable on $X \setminus D$. Let us consider the dynamical system (X, T).*

- **i)** *L_T is computable on the set of measures*
$$PM_D(X) := \{\mu \in PM(X) : \mu(D) = 0\}.$$
- **ii)** *If there is a recursively compact set of probability measures $V \subset PM(X)$ such that for every $\mu \in V$, $\mu(D) = 0$ holds, then every invariant measure isolated (for the weak topology) in V is computable.*

The precise meaning of computability on $X \setminus D$ will be given in Section 2.3. Intuitively, the meaning of the above proposition is that: if the function T is computable outside some singular set D (the discontinuity set for instance) and if we can find a set V of measures giving no weight to the set D (some class of regular measures e.g.) which contains only one invariant measure, then this measure can be computed.

We observe that in our statement we do not need any hyperbolicity assumption on the system.

As a consequence, physical measures are computable in many examples of computable systems (uniquely ergodic systems, piecewise expanding maps, systems having an indifferent fixed point and many other systems having a unique absolutely continuous invariant measure, see Theorem 3.2.1 and Propositions 3.2.2, 3.2.3).

Observe that any object which is "computable" in some way (as T, V, μ in the theorem) admits a finite description (a finite program). Item ii) in Theorem A is actually *uniform*: there is a *single* algorithm which takes finite descriptions of T and V and which, as soon as the hypothesis in Theorem A are satisfied and μ is a unique invariant measure in V, outputs a finite description of μ

(see Remark 3.2.1 and the above item b) of Problem 1). Observe that the algorithm cannot decide whether or not the hypotheses are satisfied, but computes the measure whenever they are fulfilled.

After such general statements, one could conjecture that, in computable dynamical systems, physical measures are always computable. Surprisingly, this is not true and reveals some subtlety about the general problem of computing an invariant measure. In section 4 we will see that:

Examples *There exists a computable dynamical system having no computable measure at all. Moreover, there exists a computable dynamical system on the unit interval having a single physical measure which is not computable.*

The interest of the second example comes from the fact that any computable map of the interval must have a computable fixed point, and hence a computable invariant measure. The example shows that important invariant measures can still be missed.

To further motivate these results, we finally remark that from a technical point of view, computability of the considered invariant measure is a requirement in several results about relations between computation, probability, randomness and pseudo-randomness (see e.g. [3, 19, 20, 21]).

1.1. **Plan of the paper.** In Section 2 we give a compact and self-contained introduction to the prerequisites about computable analysis which are necessary to work with dynamical systems on metric spaces, as well as some general statements about solutions of equations on metric spaces which will be used to "find" the interesting invariant measures as fixed points of the transfer operator (Theorem 2.4.3). At the end of that section we develop the computable treatment of the space of probability measures on a given (computable) metric space. Some of the results presented there are new and should be of independent interest. Their usefulness is demonstrated in the next sections.

In Section 3 we start considering dynamical systems. A direct application of the results of the previous section allows us to establish general assumptions under which the transfer operator is computable (on a suitable subset, Theorem 3.1.1).

We then use the framework and tools introduced before to face Problem 1. We prove Theorem A above (which also becomes a simple application of previous results) and show how to apply it in order to prove the computability of many interesting invariant measures.

In Section 4 we construct the two counter-examples already announced.

2. Preliminaries on algorithmic theory

2.1. **Analysis and computation.** A way to approach several problems from mathematical analysis by computational tools is to approximate the "infinite" mathematical objects (elements of non countable sets, as real numbers or functions) involved in the problem by some algorithm which constructs an approximating sequence of "finite" objects (rational numbers, polynomials with rational coefficients) which are "treatable" by the computer. Usually, the algorithm has to manipulate and decide questions about the various mathematical objects involved, and convergence results should be provided in order to choose the suitable level of accuracy for the finite approximation. The actual implementation of the algorithm and the various decisions are, in most cases, subject to round-off errors which can produce additional approximation errors, wrong decisions or undecidable situations if the error is not considered rigorously (how to decide $x \geq y$ when $x = y$ but x, y are known only up to some precision?). Sometimes, estimates for these errors can be obtained under suitable conditions, but this is in general a further and often nontrivial task (see e.g. [6]). In this paper we will work in a framework where the algorithmic abilities of the computer to represent and manipulate infinite mathematical objects are taken into account from the beginning. In this framework (often referred to as Computable Analysis) one can rigorously determine which objects can be algorithmically approximated at any given accuracy (these will be called *computable* objects), and which cannot.

Here, the word *computable* is used, but may be adapted to each particular situation: for instance, "computable" functions from $\mathbb{N}$ to $\mathbb{N}$ are called *recursive* functions, a fundamental notion of computability on subsets of $\mathbb{N}$ is that of *recursively enumerable* sets, etc.

2.2. **Background from recursion theory.** The starting point of recursion theory was to give a mathematical definition making precise the intuitive notions of *algorithmic* or *effective procedure* on symbolic objects. Every mathematician has a more or less clear intuition of what can be computed by algorithms.

Several different formalizations have been independently proposed (by Post, Church, Kleene, Turing, ...) in the 30's, and have proved to be equivalent: they compute the same functions from $\mathbb{N}$ to $\mathbb{N}$. This class of functions is now called the class of *recursive functions*. As an algorithm is allowed to run forever on an input, these functions may be *partial*, i.e. not defined everywhere. The *domain* of a recursive function is the set of inputs on which the algorithm eventually halts. A recursive function whose domain is $\mathbb{N}$ is said to be *total*. For formal definitions see for example [32].

The notion of recursive function induces directly an important computability notion on the class of subsets of $\mathbb{N}$: a set of natural numbers is said to be ***recursively enumerable*** (***r.e.*** for short) if it is the range of some partial recursive function. That is, if there exists an algorithm listing (or enumerating) the set. If the complement of a r.e. set is also r.e., then the set is said to be ***recursive***. It is easy to see that a set $E \subset \mathbb{N}$ is r.e. if and only if there is an algorithm to ***semi-decide*** whether a given integer n belongs to E or not. In other words, the algorithm halts on input n if and only if $n \in E$. Let $(E_i)_{i\in\mathbb{N}}$ be a family of r.e. subsets of $\mathbb{N}$. We say that E_i is r.e. ***uniformly in i*** if there is a single recursive function φ such that $E_i = \{\varphi(i,n) : n \in \mathbb{N}\}$. More generally, computability notions for different classes of objects (reals, open sets) will be defined in the following form:

object x is *computable* if there is a (partial or total) recursive function φ which computes x in some sense,

and for each of them, a uniform version will be implicitly defined (and intensively used in the paper) by:

objects from a family $(x_i)_{i\in\mathbb{N}}$ are *uniformly* computable if there is a single (total or partial) recursive function φ such that $\varphi(i,.)$ computes x_i for each i.

Strictly speaking, recursive functions only work on natural numbers, but this can be extended to the objects (thought of as "finite" objects) of any countable set, once a numbering of its elements has been fixed. Such a set together with its numbering will be called a ***numbered set***. For instance, the set $\mathbb{Q}$ of rational numbers can be injectively numbered $\mathbb{Q} = \{q_0, q_1, \ldots\}$ (turning it into a numbered set) in an *effective* way: the number i of a rational a/b can be computed from a and b, and vice versa. We fix such a numbering. A set of rational numbers $X \subset \mathbb{Q}$ is then r.e., if there is a r.e. set $E \subset \mathbb{N}$ such that $X = \{q_n : n \in E\}$.

The following notions were introduced by Turing in [35].

Definition 2.2.1. Let x be a real number. We say that:
- x is ***lower semi-computable*** if the set $\{q \in \mathbb{Q} : q < x\}$ is r.e.,
- x is ***upper semi-computable*** if the set $\{q \in \mathbb{Q} : q > x\}$ is r.e.,
- x is ***computable*** if it is lower and upper semi-computable. ⌟

The following classical characterization may be more intuitive: a real number is computable if and only if there exists a recursive function φ computing a sequence of rational numbers which converge exponentially to x, that is, $|q_{\varphi(i)} - x| < 2^{-i}$ for all i. We remark that, as there exist subsets of integers which are recursively enumerable but not recursive, there also exist semi-computable numbers which are not computable. In the following section we will see how these notions can be generalized to separable metric spaces, which inherit the computable structure of $\mathbb{R}$ via the metric.

2.3. **Computable metric spaces.** In this section we introduce the basic tools of computable analysis on metric spaces. Most of the results of this section and several of the following ones have been obtained by Weihrauch, Brattka, Presser and others in the framework of "Type-2 theory of Effectivity", which is based in the notion of "representation" (infinite binary codes) of mathematical objects. A standard reference book on this approach to Computable Analysis is [39], and a specific paper on computability of subsets of metric spaces is [8]. Our approach to Computable Analysis only uses the notion of recursive function (see subsection 2.2). It is intended to emphasize the fact that computability notions are just the "effective" versions of topological ones. In this way we obtain a theory syntactically familiar to most mathematicians and computability results can be proved in a transparent and compact way.

A computable metric space is a metric space with a dense numbered set such that the distance on this set is algorithmically compatible with the numbering (distances between numbered points can be computed up to arbitrary precision). From this point of view the real line (with euclidean distance) has a natural structure of computable metric space, with the rationals as a numbered set.

Definition 2.3.1. A ***computable metric space*** is a triple $\mathcal{X} = (X, d, \mathcal{S})$, where

- (X, d) is a separable metric space,
- $\mathcal{S} = \{s_i : i \in \mathbb{N}\}$ is a countable dense subset of X with a fixed numbering (the set of ***ideal points***),
- the real numbers $d(s_i, s_j)$ are all computable, uniformly in i, j.

⌟

Cantor spaces, euclidean spaces, functions spaces and manifolds with a suitable metrics can be endowed with the structure of computable metric space.

An ***ideal ball*** is a metric ball $B(s, q)$ where $s \in \mathcal{S}$ is an ideal point and q is a positive rational number. The numberings of $\mathcal{S}$ and $\mathbb{Q} \cap (0, +\infty)$ induce some canonical effective numbering $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ of the set of ideal balls.

Let $(X, d, \mathcal{S})$ be a computable metric space. The computable structure of X assures that the whole space can be "reached" using algorithmic means. Since the set $\mathcal{S}$ is dense, ideal points can approximate any point at any finite precision, and $\mathcal{B}$ is a basis of the topology.

Definition 2.3.2 (Computable points)**.** A point $x \in X$ is said to be ***computable*** if the set of ideal balls containing x is r.e. ⌟

Remark 2.3.1. As in the case of the real numbers we have the following characterization: $x \in X$ is computable if and only if there is a (total) recursive function φ such that $d(s_{\varphi(i)}, x) < 2^{-i}$ for all i. ⌟

Ideal balls are also useful to describe open sets.

Definition 2.3.3 (Recursively open sets)**.** We say that the set $U \subset X$ is ***recursively open*** if there is some r.e. set A of ideal balls such that $U = \bigcup_{B \in A} B$. That is, if there is some r.e. set $E \subseteq \mathbb{N}$ such that $U = \bigcup_{i \in E} B_i$. ⌟

Observe that the collection of r.e. open sets can be algorithmically enumerated.

Examples 2.3.1.

(1) Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of uniformly recursive open sets. The union $\bigcup_n U_n$ is a recursively open set.

(2) Let $U_1, ..., U_n$ be recursively open sets. Their intersection is recursively open also. This is a uniform operation, in the sense that there is a single algorithm which takes as input the descriptions of a finite list of open sets, and outputs the description of their intersection. $\lrcorner$

Let X, Y be two computable metric spaces. To distinguish between ideal balls of X and Y, we use the notations B_n^X, B_n^Y.

Definition 2.3.4 (Computable functions)**.** A function $T : X \to Y$ is said to be ***computable*** if $T^{-1}(B_n^Y)$ is recursively open uniformly in n. $\lrcorner$

It follows that computable functions are continuous. It is easy to see that the distance $d : X \times X \to \mathbb{R}$ is a computable function. Since we will work with functions that are not necessarily continuous everywhere, we shall consider functions that are computable on some subset of X. More precisely:

Definition 2.3.5. A function T is said to be ***computable on C*** ($C \subset X$) if there is a sequence U_n^X of uniformly recursive open sets such that

$$T^{-1}(B_n^Y) \cap C = U_n^X \cap C.$$

The set C is called the ***domain of computability*** of T. $\lrcorner$

Remark 2.3.2. Intuitively, a function T is computable (on some domain C) if there is a computer program which computes $T(x)$ (for $x \in C$) in the following sense: on input $\epsilon > 0$, the program, along its run, asks the user for approximations of x, and eventually halts and outputs an ideal point $s \in Y$ satisfying $d(T(x), s) < \epsilon$. This idea can be formalized, using for example the notion of *oracle computation*. The resulting notion coincides with the one given in the previous definitions. $\lrcorner$

As an example we show that a monotone real function whose values over the rationals are computable, is computable everywhere. This lemma will also be used later.

Lemma 2.3.1. *If $f : [0,1] \to [0,1]$ is increasing and $f(r)$ can be computed uniformly, for each rational r then f is computable.*

Proof. Let $p, q \in \mathbb{Q}$. The equality $f^{-1}(p,q) = \bigcup_{f(a) \geq p, f(b) \leq q}(a,b)$ shows that $f^{-1}((p,q))$ is recursively open, uniformly in p, q. $\square$

Let $\mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\}$ be the extended real line.

Definition 2.3.6 (Lower semi-computable functions)**.** A function $f : X \to \overline{\mathbb{R}}$ is said to be ***lower semi-computable*** if $f^{-1}(q_n, \infty)$ is recursively open uniformly in n. $\lrcorner$

It is known that there exists a recursive enumeration $\{f_i : i \in \mathbb{N}\}$ of all the nonnegative lower semi-computable functions. From the definition it follows that lower semi-computable functions are lower semi-continuous. ***Lower semi-computability on D*** is defined as for computable functions. A function f is ***upper semi-computable*** if $-f$ is lower semi-computable. It is easy to see that a real function f is computable if and only if it is upper and lower semi-computable.

Given a probability measure μ, we say that a function is ***(lower semi-) computable μ-almost everywhere*** if it is computable on a set of μ-measure one.

2.4. **Recursively compact sets and approximation of zeros.** We will give some general results about zeroes of computable functions. As in many other mathematical situations, to prove the existence of certain solutions we are helped by a suitable notion of compactness. In order for the solution to be computable, we will need a recursive version of compactness. Roughly, a compact set is recursively compact if the fact that it is covered by a finite collection of ideal balls can be tested algorithmically (for equivalence with the ϵ-net approach see Definition 2.4.2 and Proposition 2.4.4). This kind of notion and the related basic results are already present in the literature in various forms (see [8] for a complete treatment). We give a very compact self contained presentation based on the previously introduced notions.

Definition 2.4.1. A set $K \subseteq X$ is ***recursively compact*** if it is compact and there is a recursive function $\varphi : \mathbb{N}^* \to \mathbb{N}$ such that $\varphi(i_1, \dots, i_p)$ halts if and only if $(B_{i_1}, \dots, B_{i_p})$ is a covering of K. ⌟

Remark 2.4.1. Let U_i be the collection of recursively open sets (with its uniform enumeration). It is easy to see that a compact set K is recursively compact iff the set $\{i : K \subseteq U_i\}$ is r.e. ⌟

Here are some basic properties of recursively compact sets:

Proposition 2.4.1. *Let K be a recursively compact subset of X.*

(1) A singleton $\{x\}$ is recursively compact if and only if x is a computable point.
(2) $X \setminus K$ is recursively open.
(3) If U is recursively open, then $K' = K \setminus U$ is recursively compact.
(4) If K' is recursively compact then so are $K \cup K'$ and $K \cap K'$.
(5) If $f : X \to \mathbb{R}$ is lower semi-computable then so is $\inf_K f$
(6) If $f : X \to \mathbb{R}$ is upper semi-computable then so is $\sup_K f$
(7) The diameter of K is upper semi-computable.

Proof. (1) $x \in B_i \iff \{x\} \subseteq B_i$. (2) $X \setminus K = \bigcup_{K \subseteq X \setminus \overline{B}_i} B_i$ where $\overline{B}_i = \{x : d(s, x) \leq r\}$ if $B_i = B(s, r)$. (3) One has $K \setminus U \subseteq V \iff K \subseteq U \cup V$ and $U \cup V$ is recursively open uniformly in U and V. (4) $K \cup K' \subset U$ iff $K \subset U$ and $K' \subset U$. $K \cap K' = K \setminus (X \setminus K')$ and use (2) and (3). (5) $\inf_K f = \sup\{q : K \subseteq f^{-1}(q, +\infty)\}$. (6) $\sup_K f = \inf\{q : K \subseteq f^{-1}(-\infty, q)\}$. (7) Apply (6) to the computable function $d : X \times X \to \mathbb{R}$ and the recursive compact set $K \times K$.

□

Remark 2.4.2. The arguments are uniform. In point (1) for instance, this means that there is an algorithm which takes a program computing x and outputs a program testifying the recursive compactness of $\{x\}$, and vice-versa. ⌟

Corollary 2.4.1. *If $(K_i)_{i \in \mathbb{N}}$ are uniformly recursively compact sets, then so is $\bigcap_{i \in \mathbb{N}} K_i$.*

Proof. The complements of recursively compact sets are recursively open. Then by Proposition 2.4.1, part (3) the set $\bigcap_{i \in \mathbb{N}} K_i = K_0 \setminus (\bigcup_{i > 0} K_i^c)$ is recursively compact. □

It is important to remark that a recursively compact set needs not contain computable points. This will be used in section 4.

Proposition 2.4.2. *There exists a nonempty recursively compact set $K \subset [0, 1]$ containing no computable point.*

Proof. Let I_n be an enumeration of all the rational intervals and $\epsilon > 0$ be a rational number. Consider an enumeration φ_i of all the partial recursive functions. Put $E = \{i \geq 1 : \varphi_i(i) \text{ halts and } |I_{\varphi_i(i)}| < \epsilon 2^{-i}\}$. E is a r.e. subset of $\mathbb{N}$. Let $U = \bigcup_{i \in E} I_i$: $\lambda(U) \leq \sum_{i \in E} \epsilon 2^{-i} \leq \epsilon$. Let $x \in [0, 1]$ be a computable real number. There is a total recursive function φ_i such that $|I_{\varphi_i(n)}| < \epsilon 2^{-n}$ and $x \in I_{\varphi_i(n)}$ for all n, so $i \in E$ and $x \in U$. Hence U contains all computable points. As $[0, 1]$ is recursively compact, so is $K = [0, 1] \setminus U$. □

The following proposition is an elementary example of how many statements about topology and calculus on metric spaces can be easily translated to the computable setting.

Proposition 2.4.3 (Stability by computable functions)**.** *Let $f : K \subseteq X \to Y$ be a function computable on a recursively compact set K. Then $f(K)$ is recursively compact.*

Proof. Let U_n^Y be an enumeration of all the recursively open sets of Y. As f is computable on K, there are uniformly recursively open sets $V_n \subseteq X$ such that $f^{-1}(U_n^Y) \cap K = V_n \cap K$. The set $f(K)$ is recursively compact because the relation $f(K) \subseteq U_n^Y$ is semi-decidable. Indeed, $f(K) \subseteq U_n^Y \iff K \subseteq f^{-1}(U_n^Y) \iff K \subseteq V_n$. □

Remark that the argument is uniform: if $(K_i)_{i\in\mathbb{N}}$ is a sequence of uniformly recursively compact subsets of X on which f is computable, then $(f(K_i))_{i\in\mathbb{N}}$ is a sequence of uniformly recursively compact subsets of Y. We will say that $f(K)$ is recursively compact *uniformly in K*.

As a first simple example of application, we observe that in some cases the global attractor of a (computable) dynamical system can be approximated by an algorithm.

Corollary 2.4.2. *Let X be a recursively compact computable metric space and T a computable dynamics on it. Then the set:*

$$\Lambda := \bigcap_{n\geq 0} T^n(X)$$

is recursively compact.

Proof. Apply Proposition 2.4.3 and Corollary 2.4.1 □

As said before, we will compute invariant measures by approximating the fixed points of the tranfer operator. There is a well known result stating computability of isolated zeros (and then fixed points) of computable functions (see [38], and [12]). We will need the following more general version, applicable to any metric space. Observe that in this version, the function need not be everywhere computable, but only on some compact subset where we are looking for the solution.

Theorem 2.4.1. *Let K be a recursively compact subset of X and $f : X \to \mathbb{R}$ be a function computable on K. Then every isolated (in K) zero of f is computable.*

Proof. Let x_0 be an isolated zero of f. Let $B(s,r)$ be an ideal ball containing $x_0$such that the only zero of f lying in $\overline{B}(s,r) \cap K$ is x_0. The set $N = \{x : f(x) \neq 0\} \cup \{x : d(x,s) > r\}$ is recursively open in K (that is, $N \cap K = U \cap K$ with U recursively open), so $\{x_0\} = K \setminus N = K \setminus U$ is recursively compact by Proposition 2.4.1. Hence, x_0 is a computable point. □

Remark 2.4.3. Observe that the argument is uniform in f and an ideal ball isolating the zero (if the zero is unique in K the ball is not needed). In particular, there is an algorithm which takes a finite description of f and the ball and outputs the corresponding zero. ⌟

Corollary 2.4.3. *Let K be a recursively compact subset of X and $f : X \to X$ be a function computable on K. Then every isolated (in K) fixed point of f is computable.*

Proof. Apply the preceding theorem to the function $g : X \to \mathbb{R}$ defined by $g(x) = d(x, f(x))$. □

We will use the following characterization of recursive compactness, by means of effective ϵ-nets.

Definition 2.4.2. A computable metric space is ***recursively precompact*** if there is a total recursive function $\varphi : \mathbb{N} \to \mathbb{N}^*$ such that for all n, $\varphi(n)$ computes a 2^{-n}-net: that is $\varphi(n) = (i_1, \ldots, i_p)$ where $(s_{i_1}, \ldots, s_{i_p})$ is a 2^{-n}-net. ⌟

Here is a computable version of a classical theorem:

Proposition 2.4.4. *Let X be a computable metric space. X is recursively compact if and only if it is complete and recursively precompact.*

Proof. If X is recursively compact then we define the following algorithm: it takes n as input, then enumerates all the $(i_1, \ldots, i_p)$, and tests whether $(B(s_{i_1}, 2^{-n}), \ldots, B(s_{i_p}, 2^{-n}))$ is a covering of X (this is possible by recursive compactness). As X is compact, hence precompact, such a covering exists and will be eventually enumerated: output it. The algorithm makes X recursively precompact.

Suppose that X is complete and recursively precompact. Let $(B(s_1,q_1),\dots,B(s_k,q_k))$ be ideal balls: we claim that $(B(s_1,q_1),\dots,B(s_k,q_k))$ covers X if and only if there exists n such that each point s of the 2^{-n}-net given by recursive precompactness lies in a ball $B(s_i,q_i)$ satisfying $d(s,s_i)+2^{-n}<q_i$. The procedure which enumerates all the n and semi-decides this halts if and only if the initial sequence of balls covers X. We leave the proof of the claim to the reader (take n such that 2^{-n} is less than the Lebesgue number of the finite covering). □

2.5. **Computable measures.** Let us consider the space $PM(X)$ of Borel probability measures over X. Let $C_0(X)$ be the set of real-valued bounded continuous functions on X. We recall the notion of weak convergence of measures:

Definition 2.5.1. μ_n is said to be ***weakly convergent*** to μ if $\int f\,\mathrm{d}\mu_n \to \int f\,\mathrm{d}\mu$ for each $f\in C_0(X)$. ⌟

Let us introduce the Wasserstein-Kantorovich distance between measures. Let μ_1 and μ_2 be two probability measures on X and consider:

$$W_1(\mu_1,\mu_2)=\sup_{f\in 1\text{-Lip}(X)}\left|\int f\,\mathrm{d}\mu_1-\int f\,\mathrm{d}\mu_2\right|$$

where 1-Lip(X) is the space of functions on X having Lipschitz constant less than one. We remark that since adding a constant to the test function f does not change the above difference $\int f\,\mathrm{d}\mu_1-\int f\,\mathrm{d}\mu_2$, the supremum can be taken over the set of 1-Lipschitz functions mapping a distinguished ideal point s_0 to 0. The distance W_1 has the following useful properties which will be used in the following.

Proposition 2.5.1 (see [1] Prop 7.1.5)**.**

(1) W_1 is a distance and if X is bounded, separable and complete, then $PM(X)$ with this distance is a separable and complete metric space.
(2) If X is bounded, a sequence is convergent for the W_1 metrics if and only if it is convergent for the weak topology.
(3) If X is compact $PM(X)$ is compact with this topology.

Item (1) has an effective version: $PM(X)$ inherits the computable metric structure of X. Indeed, given the set $\mathcal{S}_X$ of ideal points of X we can naturally define a set of ideal points $\mathcal{S}_{PM(X)}$ in $PM(X)$ by considering finite rational convex combinations of the Dirac measures δ_s supported on ideal points $s\in S_X$. This is a dense subset of $PM(X)$. The proof of the following proposition can be found in ([24]).

Proposition 2.5.2. *If X bounded then $(PM(X),W_1,\mathcal{S}_{PM(X)})$ is a computable metric space.*

A measure μ is then computable if there is a sequence $\mu_n\in\mathcal{S}_{PM(X)}$ converging exponentially fast to μ (see Remark 2.3.1) in the W_1 metric (and hence for the weak convergence).

The following lemma, which will be very important to us, says that point (3) of Proposition 2.5.1 also has an effective version:

Lemma 2.5.1. *If X is a recursively precompact metric space, then $PM(X)$ with the W_1 distance is a recursively precompact metric space.*

Proof. We will show how to effectively find an $r-$net for each r of the form $r=\frac{1}{n}, n\in\mathbb{N}$. Let us consider the set $S_r=\{\frac{k}{n},0\le k\le n\}$ subdividing the unit intervals in equal segments. Let us also consider an r-net $N_r = \{x_1, ...x_m\}$ constructed by recursive compactness of X. Now let us consider the set Υ_r of measures with support in N_r given by

$$\Upsilon_r = \{k_1\delta_{x_1} + ... + k_m\delta_{x_m}\ s.t.\ k_i \in S_r\ ,\ k_1 + ... + k_m = 1\}.$$

This is a $2r$ net in $PM(X)$. To see this let us consider a probability measure μ on X and a ball $B(x_1, r)$ centered in $x_1 \in X$. Let us consider the measure μ_1 defined by

$$\mu_1(A) = \mu(A) - \mu(B(x_1, r) \cap A) + \mu(B(x_1, r))\delta_{x_1}(A)$$

for each measurable set $A \subset X$. The measure μ_1 is obtained transporting the mass contained in the ball $B(x_1, r)$ to its center. Then $W_1(\mu_1, \mu) \leq r\mu(B(x_1, r))$. Let us now consider the sequence of measures $\mu_1, ..., \mu_m$ where μ_1 is as before and the other ones are given by

$$\mu_i(A) = \mu_{i-1}(A) - \mu_{i-1}(B(x_i, r) \cap A) + \mu(B(x_i, r))\delta_{x_i}(A),$$

at the end μ_m is a measure with support in N_r and by the triangle inequality $W_1(\mu_m, \mu) \leq r$.

Now μ_m has the same support as the measures in Υ_r and there is $\nu \in \Upsilon_r$ such that $|\int f\,\mathrm{d}\mu_n - \int f\,\mathrm{d}\nu| \leq r$ (recall that $r = 1/n$) for each $f \in$ 1-Lip(X), hence $W_1(\mu_m, \nu) \leq r$ and then $W_1(\mu, \nu) \leq 2r$ and this proves the statement. □

We recall that the nonnegative lower semi-computable functions on X can be recursively enumerated: let f_i be such an enumeration. The computability of a measure can be characterized this way (see [24], Corollary 4.3.1):

Lemma 2.5.2. *Let X be a bounded computable metric space and C be any subset of $PM(X)$.*

(1) the functions $\mu \mapsto \int f_i\,\mathrm{d}\mu$ are lower semi-computable, uniformly in i,
(2) $\mu \in PM(X)$ is computable iff the numbers $\int f_i\,\mathrm{d}\mu$ are lower semi-computable, uniformly in i,
(3) $L : PM(X) \to PM(X)$ is computable on C iff the functions $\mu \mapsto \int f_i\,\mathrm{d}L(\mu)$ are lower semi-computable on C, uniformly in i.

An interesting remark about computable measures is that they must have computable points in the support. This will be used in section 4.1.

Proposition 2.5.3. *If μ is a computable probability measure, then there exist computable points in the support of μ.*

Proof. The sequence of functions $f_i := 1_{B_i}$ (the indicator functions of ideal balls) are uniformly lower semi-computable. By Lemma 2.5.2, the numbers $\int f_i\,\mathrm{d}\mu = \mu(B_i)$ are uniformly lower semi-computable. Hence, the set $E = \{B_i : B_i \cap \mathrm{supp}(\mu) \neq \emptyset\} = \{B_i : \mu(B_i) > 0\}$ is recursively enumerable. From any ideal ball $B = B(s, q) \in E$, we can effectively construct a decreasing sequence of ideal closed balls intersecting supp(μ), and whose radius decrease exponentially fast to zero. Their intersection is a singleton that contains a computable point. □

3. Computing invariant measures

Let X be a metric space, $T : X \mapsto X$ a Borel measurable map and μ a T-invariant Borel probability measure. A set A is called T-invariant if $T^{-1}(A) = A$ (mod 0). The system (X, T, μ) is said to be ergodic if each T-invariant set has total or null measure. In such systems the famous Birkhoff ergodic theorem says that time averages computed along μ-typical orbits coincides with space average with respect to μ. More precisely, for any $f \in L^1(X, \mu)$ it holds

$$\lim_{n\to\infty} \frac{S_n^f(x)}{n} = \int f\,\mathrm{d}\mu, \tag{3.1}$$

for μ almost each x, where $S_n^f = f + f \circ T + \ldots + f \circ T^{n-1}$.

This shows that in an ergodic system, the statistical behavior of observables, under typical realizations of the system is given by the average of the observable made with the invariant measure.

We say that a point x belongs to the basin of an invariant measure μ if (3.1) holds at x for each bounded continuous f. In case X is a manifold (possibly with boundary), a physical measure is an invariant measure whose basin has positive Lebesgue measure (for more details and a general survey see [37]).

In the applied literature the most common method to simulate or understand the above statistical behaviors is to compute and study some trajectory. This method has three main theoretical problems which motivate the search of another approach:

- numerical errors,
- typicality of the sample,
- how many iteration are necessary?

The first (and widely known) problem is the amplification of the numerical error (if the system is sensitive to initial conditions as most interesting systems are). Here the shadowing results are often invoked to justify the correctness of simulations, but rigorous results are proved only for a small class of systems (see e.g. [30]) and moreover the mere existence of a shadowing orbit does not say anything about its typicality (see e.g. [6, 7] for a further discussion on numerical errors).

The second problem is indeed that this method should compute, in order to be useful, a trajectory which shows the "typical" behavior of the system: a behavior which takes place with large or full probability. The main problem here is the fact that the set of initial conditions the computer has access to, being countable, has probability zero. Hence, there is no guarantee that what we see on the screen is typical in some sense. On the contrary, in a chaotic system, typical orbits are far from being describable by a finite program. It is true for example that in an ergodic system having positive entropy h, a typical n-steps orbit segment needs approximatively a program which is hn bits long to be described (up some approximation ϵ, see e.g. [10] for the original result or [18, 21] for a version in the framework of computable analysis). We remark, however, that if one looks for points which behave as typical ones for Birkhoff averages (hence they look typical with respect to some particular aspect) there are some rigorous results partly supporting this way to proceed: in several classes of systems there are computable initial conditions which are typical with respect to Birkhoff averages (see [20] for a precise result).

The third problem however remains. Even if one finds a program describing a typical orbit of the system, how many iterations should be considered to be close to the limit behavior, so that the orbit represents the invariant measure up to a certain approximation? Although this problem can be approached rigorously in some cases (see [11, 2] e.g.) we will not adopt this point of view. We will study the system's statistical behavior by directly computing the invariant measure as fixed points of a certain transfer operator.

3.1. **The transfer operator.** A function T between metric spaces naturally induces a linear function L_T between probability measure spaces. This function L_T is linear and is called transfer operator (associated to T). Measures which are invariant for T are fixed points of L_T.

Let us consider a computable metric space X and a measurable function $T : X \to X$. Let us also consider the space $PM(X)$ of Borel probability measures on X.

Let us define the function $L_T : PM(X) \to PM(X)$ by duality in the following way: if $\mu \in PM(X)$ then $L_T(\mu)$ is such that

$$\int f \, \mathrm{d}L_T(\mu) = \int f \circ T \, \mathrm{d}\mu$$

for each $f \in C(X)$. In the next sections, invariant measures will be found as solutions of the equation $W_1(\mu, L(\mu)) = 0$. To apply Theorem 2.4.1 and Corollary 2.4.3 to this equation we need that L be computable. We remark that if T is not continuous then L is not necessarily continuous (this can be realized by applying L to some delta measure placed near a discontinuity point) hence not computable. Still, we have that L is continuous (and its modulus of continuity is computable) at all measures μ which are "far enough" from the discontinuity set D. This is technically expressed by the condition $\mu(D) = 0$.

Theorem 3.1.1. *Let X be a computable metric space and $T : X \to X$ be a function which is computable on $X \setminus D$. Then L_T is computable on the set of measures*

$$PM_D(X) := \{\mu \in PM(X) : \mu(D) = 0\}. \tag{3.2}$$

Proof. By Lemma 2.5.2, we have to prove that the function $\mu \mapsto \int f_i \, \mathrm{d}L_T(\mu) = \int f_i \circ T \, \mathrm{d}\mu$ is lower semi-computable on $PM_D(X)$, uniformly in i. The difficulty here comes from the fact that the functions $f_i \circ T$ are not lower semi-computable on the whole space anymore, but only on $X \setminus D$. We overcome this way: for each i, we construct a function g_i, which is lower semi-computable on the whole space, and that coincides with $f_i \circ T$ on $X \setminus D$. The construction is as follows: let U_n be uniformly recursively open sets such that $(f_i \circ T)^{-1}(q_n, \infty) \setminus D = U_n \setminus D$ (q_n is the enumeration of the rationals). The function g_i is defined by

$$g_i(x) = \sup_n q_n 1_{U_n}(x).$$

The function $\mu \mapsto \int g_i \, \mathrm{d}\mu$ is lower semi-computable, uniformly in i, by Lemma 2.5.2, item (1). For $\mu \in PM_D(X)$, $\int g_i \, \mathrm{d}\mu = \int f_i \circ T \, \mathrm{d}\mu$ so L_T is computable on $PM_D(X)$, by Lemma 2.5.2, item (3). □

In particular, if T is computable on the whole space X then L is computable on all $PM(X)$.

3.2. **Computing invariant "regular" measures.** The above tools allow us to ensure the computability of L_T on a large class of measures. This will enable us to apply Corollary 2.4.3 and see an invariant measure as a fixed point.

Theorem 3.2.1. *Let X be a computable metric space and T a function which is computable on $X \setminus D$. Suppose there is a recursively compact set of probability measures $V \subset PM(X)$ such that for every $\mu \in V$, $\mu(D) = 0$ holds. Then every invariant measure isolated in V is computable.*

Proof. By Theorem 3.1.1, L_T is computable on V. Since V is recursively compact, Theorem 2.4.1 implies the computability of any fixed point μ of L_T, i.e. any T-invariant measure, that is isolated in V. □

Remark 3.2.1. This theorem is uniform: there is an algorithm which takes as inputs finite descriptions of T, V and an ideal ball in $M(X)$ which isolates an invariant measure μ, and outputs a finite description of μ (see the above proof and Remark 2.4.3). ⌟

A trivial and general consequence of Theorem 3.2.1 is the following:

Corollary 3.2.1. *If a system as above is uniquely ergodic and its invariant measure μ satisfies $\mu(D) = 0$, then it is a computable measure.*

The main problem in the application of Theorem 3.2.1 is the requirement that the invariant measure we are trying to compute be isolated in V. In general the space of invariant measures in a given dynamical system could be very large (an infinite dimensional convex subset of $PM(X)$). To isolate a particular measure we can restrict and consider a subclass of "regular" measures.

Let us consider the following *seminorm*:

$$\|\mu\|_\alpha = \sup_{x \in X, r > 0} \frac{\mu(B(x, r))}{r^\alpha}.$$

Proposition 3.2.1. *If α and K are computable and X is recursively compact then*

$$V_{\alpha,K} = \{\mu \in PM(X) : \|\mu\|_\alpha \le K\} \tag{3.3}$$

is recursively compact.

Proof. $U = \{\mu \in PM(X) : \|\mu\|_\alpha > K\}$ is recursively open. Indeed, $\|\mu\|_\alpha > K$ iff there exists $s, r \in \mathcal{S} \times \mathbb{Q}$ such that $\mu(B(s,r)) > Kr^\alpha$. As $\mu \mapsto \mu(B(s,r))$ is lower semi-computable uniformly in s, r, the sets $U_{s,r} := \{\mu : \mu(B(s,r)) > Kr^\alpha\}$ are uniformly recursively open subsets of $PM(X)$. Hence, $U = \cup_{s,r} U_{s,r}$ is recursively open.

Now, $V_{\alpha,K} = PM(X) \setminus U$. As $PM(X)$ is recursively compact by Lemma 2.5.1 and Proposition 2.4.4, and U is recursively open, then Proposition 2.4.1 item (3) allows us to conclude. □

In Theorem 3.2.1 we require that $\mu(D) = 0$ hold. This is automatically true in many examples when the measure is regular and the set D is reasonably small (we denote by dim_H the Hausdorff dimension).

Proposition 3.2.2. *Let X be recursively compact and T be computable on $X \setminus D$, with $dim_H(D) < \infty$. Then any invariant measure isolated in $V_{\alpha,K}$ with $\alpha > dim_H(D)$ is computable.*

Proof. Let us first prove that $\mu(D) = 0$ for all $\mu \in V_{\alpha,K}$. For all $\epsilon > 0$, there is a covering $(B(x_i, r_i))_i$ of D satisfying $\sum_i r_i^\alpha < \epsilon$. Hence $\mu(D) \le \sum_i \mu B(x_i, r_i) \le 2^\alpha K \sum_i r_i^\alpha \le 2^\alpha K\epsilon$. As this is true for each $\epsilon > 0$, $\mu(D) = 0$.

The result then follows from the fact that $V_{\alpha,K}$ is recursively compact and Theorem 3.2.1. □

Remark 3.2.2. Once again, this is uniform in T, α, K. ⌟

The above general propositions allow us to obtain as a corollary the computability of many absolutely continuous invariant measures. For the sake of simplicity, let us consider maps on the interval.

Proposition 3.2.3. *If $X = [0,1]$, T is computable on $X \setminus D$, with $dim_H(D) < 1$ and (X,T) has a unique absolutely continuous invariant measure μ with bounded density, then μ is computable.*

Proof. The result follows from Proposition 3.2.2 applied to $\alpha = 1$ and some $K \ge \|f\|_{L^\infty}$ where f is the density of μ. Indeed, μ is absolutely continuous with bounded density if and only if $\|\mu\|_1 < \infty$, and in that case $\|\mu\|_1 = \|f\|_{L^\infty}$, so $V_{1,K}$ contains only one invariant measure.

Let us prove that μ is absolutely continuous with bounded density if $\|\mu\|_1 < \infty$. Let $l > \|\mu\|_1$. Let us consider the conditional expectation $E[\mu|I^n]$ of μ to the dyadic n-th grid $I^n = \{[k2^{-n}, (k+1)2^{-n}), 0 \le k \le 2^n\}$.

Since $\|\mu\|_1 = l$, this a fortiori implies $0 \le E[\mu|I^n] \le l$ a.e. By the Doob's martingale convergence theorem it follows that $E[\mu|I^n]$ has an a.e. pointwise and L^1 limit f and $f \le l$ a.e.. Since f is bounded then it is a density for μ.

□

d-dimensional submanifolds of $\mathbb{R}^n$ can naturally be endowed with a structure of computable metric spaces (see [20]). Considering a dyadic grid on $\mathbb{R}^d$ and chart diffeomorphisms it is straightforward to prove, in the same way as before:

Corollary 3.2.2. *Let X be a recursively compact d dimensional C^1 submanifold of $\mathbb{R}^n$ (with or without boundary). If T is computable on $X \setminus D$, with $dim_H(D) < d$ and (X,T) has a unique absolutely continuous invariant measure μ with bounded density, then μ is computable.*

As it is well known, interesting examples of systems having a unique absolutely continuous invariant measure (with bounded density as required) are topologically transitive *piecewise expanding maps* on the interval or *expanding maps* on manifolds (see [36] for precise definitions). Provided that the dynamics is computable we then have by the above propositions that the absolutely continuous invariant measure is computable too.

3.3. **Unbounded densities.** The above results ensure computability of some absolutely continuous invariant measure with bounded density. If we are interested in situations where the density is unbounded, we can consider a new norm, "killing" singularities.

Let us hence consider a computable function $f : X \to \mathbb{R}$ and

$$\|\mu\|_{f,\alpha} = \sup_{x \in X, r>0} \frac{f(x)\mu(B(x,r))}{r^\alpha}.$$

Propositions 3.2.1 and 3.2.2 also hold for this norm. If f is such that $f(x) = 0$ when $\lim_{r\to 0} \frac{\mu(B(x,r))}{r^\alpha} = \infty$ this can let the norm be finite when the density diverges.

As an example, where this can be applied, let us consider the Manneville Pomeau type maps on the unit interval. These are maps of the type $x \to x + x^z (\text{mod } 1)$. When $1 < z < 2$ the dynamics has a unique absolutely continuous invariant measure μ_z having density $e_z(x)$ which diverges at the origin as $e_z(x) \asymp x^{-z+1}$ and is bounded elsewhere (see [26] Section 10 and [36] Section 3 e.g.). If we consider the norm $\|.\|_{f,1}$ with $f(x) = x^2$ we have that $\|\mu_z\|_{f,1}$ is finite for each such z. By this it follows that for each such z the measure μ_z is computable.

4. Computable systems with uncomputable invariant measures

We have seen that the technique presented above proves the computability of many absolutely continuous invariant measure which are also physical measures. As we have seen in the introduction, with other techniques it is possible to prove the computability of other physical measures (axiom A systems e.g., see [20]). This raises naturally the following question: a computable systems does necessarily have a computable invariant measure? what about ergodic physical measures?

The following is an easy example showing that this is not true in general even in quite regular systems, hence the whole question of computing invariant measures has some subtlety.

Let us consider a system on the unit interval given as follows. Let $\tau \in (0,1)$ be a lower semi-computable real number which is not computable. There is a computable sequence of rational numbers τ_i such that $\sup_i \tau_i = \tau$. For each i, define $T_i(x) = \max(x, \tau_i)$ and $T(x) = \sum_{i\geq 1} 2^{-i} T_i$. The functions T_i are uniformly computable so T is also computable.

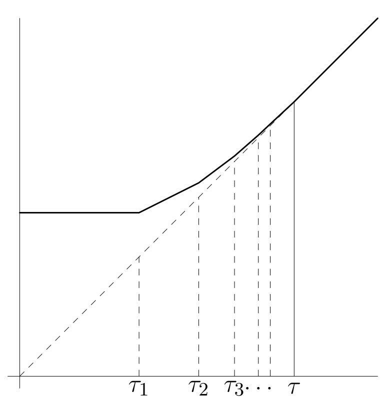

Figure 1. The map T.

The system $([0,1], T)$ is hence a computable dynamical system. T is non-decreasing, and $T(x) > x$ if and only if $x < \tau$. This system has a physical ergodic invariant measure which is δ_τ, the Dirac measure placed on τ. The measure is physical because τ attracts all the interval at its left.

Since τ is not computable then δ_τ is not computable. We remark that coherently with the previous theorems δ_τ is not isolated.

It is easy to prove, by a simple dichotomy argument, that a computable function from $[0,1]$ to itself must have a computable fixed point. Hence it is not possible to construct a system over the interval having no computable invariant measure (we always have the δ over the fixed point). With some more work we will see that such an example can be constructed on the circle.

4.1. **A computable system without computable invariant measures.** We go further and exhibit a computable dynamical system on a compact space which has *no* computable invariant probability measure.

We consider the unit circle S, identified with $\mathbb{R}/\mathbb{Z}$. It naturally has a computable metric structure inherited from that of $\mathbb{R}$. As said before, on S, there is a computable map with no computable invariant probability measure. We construct such a map $T : [0,1] \to \mathbb{R}$ satisfying $T(1) = T(0) + 1$, and consider its quotient on the unit circle.

From Proposition 2.4.2 we know that there is a non-empty recursively compact set K containing no computable point. Let $U = (0,1) \setminus K$: this is a r.e. open set, so there are computable sequences a_i, b_i ($i \geq 1$) such that $0 < a_i < b_i < 1$ and $U = \bigcup_i (a_i, b_i)$. Let us define non-decreasing, uniformly computable functions $f_i : [0,1] \to [0,1]$ such that $f_i(x) > x$ if $x \in (a_i, b_i)$ and $f_i(x) = x$ otherwise. For instance, $f_i(x) = 2x - a_i$ on $[a_i, \frac{a_i+b_i}{2}]$ and $f_i(x) = b_i$ on $[\frac{a_i+b_i}{2}, b_i]$.

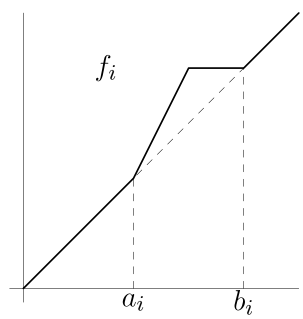

FIGURE 2. The map f_i.

As neither 0 nor 1 belongs to K, there is a rational number $\epsilon > 0$ such that $K \subseteq [\epsilon, 1-\epsilon]$. Let us define $f : [0,1] \to \mathbb{R}$ by $f(x) = x$ on $[\epsilon, 1-\epsilon]$, $f(x) = 2x - (1-\epsilon)$ on $[1-\epsilon, 1]$ and $f(x) = \epsilon$ on $[0, \epsilon]$.

We then define the map $T : [0,1] \to \mathbb{R}$ by $T(x) = \frac{f}{2} + \sum_{i \geq 2} 2^{-i} f_i$. T is computable and non-decreasing, and $T(x) > x$ if and only if $x \in [0,1] \setminus K$. As $T(1) = 1 + \frac{\epsilon}{2} = 1 + T(0)$, we can take the quotient of T modulo 1.

Proposition 4.1.1. *$W = U \cup [0, \epsilon) \cup (1-\epsilon, 1]$ is a strictly invariant set: $T^{-1}W = W$.*

Proof. If $x \notin W$ then $T(x) = x \notin W$.

If $x \in W$ then $T(x) \in W$. Indeed, if $T(x) \notin W$, $T(x)$ is a fixed point so T is constant on $[x, T(x)]$ (T is non-decreasing). Let q be any rational number in $(x, T(x))$: $T(x) = T(q)$ is then computable, but does not belong to W: impossible. □

Proposition 4.1.2. *The map T is computable but has no computable invariant probability measure.*

Let $x \in [0,1]$: the trajectory of x is "non-decreasing" and converges to the first point above x which is not in U, $\inf([x,1] \setminus U)$ or to $\min(K)$ if $x > \sup(K)$. More precisely, there are two cases: (i) if $x \notin U$ then x is a fixed point (unstable on the right), (ii) if $x \in U$ then the trajectory of x converges to a lower semi-computable fixed point (non-computable, as it does not belong to U).

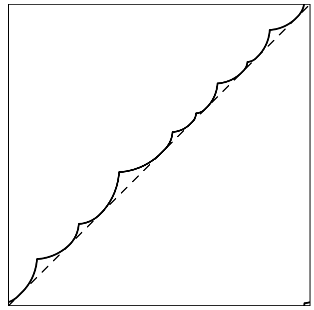

FIGURE 3. The map T.

Lemma 4.1.1. *Let μ be an invariant probability measure: then $\mu(K^c) = 0$.*

Proof. Obviously $\mu(0) = 0$ because 0 is not periodic. Let $(a, b) = (a_i, b_i)$ be an interval from the description of U. Since $T^n(a)$ and $T^n(b)$ tends to some non computable α (and then are not stationary, as they are computable), the interval (a, b) is wandering. Hence, by Poincaré recurrence theorem it has null measure. □

Proof. (of proposition 4.1.2) We can conclude: let μ be a computable invariant probability measure: by the above lemma its support is then included in the complement of W. But the support of a computable probability measure always contains computable points (see proposition 2.5.3) : contradiction. □

Actually, the set of invariant measures is exactly the set of measures which give null weight to W. It is easy to see that in the above system the set of invariant measures is a convex recursive compact set. Indeed, the function $\mu \to \mu(W)$ is lower semi-computable, so $\{\mu : \mu(W) > 0\}$ is a recursive open set. Its complement is then a recursive compact set, as the whole space of probability measures is a recursive compact set. The above example hence shows an example of a convex, and recursive compact set whose extremal points are not computable. We also notice that with a slightly modification of the various f_i (see Fig. 2) it is possible to give also a smooth system having the same properties as the examples in this section.

We end by remarking that the construction we have presented here has been adapted by Stephen Simpson (unpublished) to prove the following theorem in reverse mathematics:

Theorem 4.1.1. WKL_0 *is equivalent over* RCA_0 *to the statement that for every self-homeomorphism of a compact metric space there exists an invariant probability measure.*

(See also the WKL_0 version of the Schauder Fixed Point Theorem, [33, Theorem IV.7.9].)

REFERENCES

[1] L. Ambrosio, N. Gigli, G. Savare. *Gradient flows: in metric spaces and in the space of probability measures*, Birkhauser Zurich 2005 ebnisse der Mathematik und ihrer Grenzgebiete. 10

[2] Jeremy Avigad, Philipp Gerhardy, and Henry Towsner. Local stability of ergodic averages. *Transactions of the American Mathematical Society*, 362:261–288, 2010. 12

[3] L. Bienvenu, W. Merkle. *Effective Randomness for Computable Probability Measures* Electr. Notes Theor. Comput. Sci. vol 167, (2007) p. 117-130. 4

[4] I. Binder, M. Braverman, M. Yampolsky. *Filled Julia sets with empty interior are computable*, Journal FoCM, 7(2007), 405-416.

[5] I. Binder, M. Braverman, M. Yampolsky. *On computational complexity of Siegel Julia sets*, Commun. Math. Phys., 264, 317-334(2006)

[6] M. L. Blank. *Pathologies generated by round-off in dynamical systems.* Physica D. (1994) vol 78, no 1-2, pp: 93–114. 4, 12

[7] M. L. Blank. *Small perturbations of chaotic dynamical systems*. Russian Mathematical Surveys. (1989) vol 44, no 6, pp:1–33. 12

[8] Vasco Brattka and Gero Presser. Computability on subsets of metric spaces. *Theoretical Computer Science*, 305(1-3):43–76, 2003. 6, 8

[9] M. Braverman, M. Yampolsky. *Non-computable Julia sets*, Journ. Amer. Math. Soc., 19 (2006), 551-578.

[10] A. A. Brudno (1983) *Entropy and the complexity of the trajectories of a dynamical system*. Trans. Mosc. Math. Soc. **44** 127–151. 12

[11] J-R. Chazottes, P. Collet and B. Schmitt. *Statistical consequences of the Devroye inequality for processes. Applications to a class of non-uniformly hyperbolic dynamical systems* 2005 Nonlinearity 18 2341-2364. 12

[12] P. Collins. Computability and Representations of the Zero Set. *Theoretical Computer Science*, 221:37–43, 2008. 9

[13] M. Dellnitz, A. Hohmann. *A subdivision algorithm for the computation of unstable manifolds and global attractors*. Numerische Mathematik. 1997, vol. 75, no3, pp. 293-317.

[14] M. Dellnitz, O. Junge. *On the Approximation of Complicated Dynamical Behavior*. SIAM Journal on Numerical Analysis. 1999, vol. 36, no2, pp. 491-515.

[15] J. Ding, Q. Du, T. Y. Li. *High order approximation of the Frobenius-Perron operator*. Applied Mathematics and Computation. (1993) vol 53, pp: 151 - 171. 2

[16] J. Ding, A. Zhou. *The projection method for computing multidimensional absolutely continuous invariant measures*. Journal of Statistical Physics.(1994) vol 77, 3-4, pp: 899-908. 2

[17] P. Gács. *Lectures notes on descriptional complexity and randomness*. Boston University (1993) 1–67.

[18] S. Galatolo. *Orbit complexity by computable structures* Nonlinearity **13**, 1531-1546 (2000). 12

[19] P. Gács, M. Hoyrup, C. Rojas. Randomness on computable probability spaces - a dynamical point of view. In Susanne Albers and Jean-Yves Marion, editors, *26th International Symposium on Theoretical Aspects of Computer Science (STACS 2009)*, pages 469–480, Dagstuhl, Germany, 2009. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, Germany. 4

[20] S. Galatolo, M. Hoyrup, and C. Rojas. A constructive Borel-Cantelli lemma. Constructing orbits with required statistical properties. *Theor. Comput. Sci.*, 410(21-23):2207–2222, 2009. 3, 4, 12, 14, 15

[21] S. Galatolo, M. Hoyrup, and C. Rojas. Effective symbolic dynamics, random points, statistical behavior, complexity and entropy. *Inf. Comput.*, 208(1):23–41, 2010. 4, 12

[22] S. Galatolo, M. J. Pacifico. *Lorenz like flows: exponential decay of correlations for the Poincaré map, logarithm law, quantitative recurrence.* http://front.math.ucdavis.edu/0901.0574 3

[23] P. Hertling *Is the Mandelbrot set computable*, Math. Logic Quart, 2005, vol 51, pp. 5–18.

[24] M. Hoyrup and C. Rojas. Computability of probability measures and Martin-Löf randomness over metric spaces. *Inf. Comput.*, 207(7):830–847, 2009. 10, 11

[25] B. Hunt. *Estimating invariant measures and Lyapunov exponents* preprint - http://www.chaos.umd.edu/~bhunt/research/eimale.pdf (1995) 2

[26] S. Isola. *On systems with finite ergodic degree.* Far East Journal of Dynamical Systems 5(2003), 1-62 15

[27] M. Keane, R. Murray, L. S. Young. *Computing invariant measures for expanding circle maps* Nonlinearity 11 27-46 (1998) 2

[28] Y. Kifer. *General random perturbations of hyperbolic and expanding transformations*. Journal d'Analyse Mathématique. (1986) vol 47, 1, pp: 111-150.

[29] C. Liverani. *Rigorous numerical investigations of the statistical properties of piecewise expanding maps-A feasibility study*. Nonlinearity 14, n.3 pp. 463-490, (2001). 2

[30] K. Palmer. *Shadowing in Dynamical Systems - Theory and Applications*. Mathematics and Applications vol 501. Springer (2000). 12

[31] M. Pollicott, O. Jenkinson. *Computing Invariant Densities and Metric Entropy* Comm. Math. Phys. 211, n. 3 (2000) 687-703 2

[32] H. Rogers. *Theory of recursive functions and effective computability* MIT Press Cambridge, MA, USA (1987) 5

[33] S. G. Simpson. *Subsystems of Second Order Arithmetic*. Perspectives in Mathematical Logic, Springer-Verlag. Cambridge University Press (1999). 17

[34] W. Tucker. *The Lorenz attractor exists*, C. R. Acad. Sci. Paris, 1999 vol 328, pp. 1197–1202.

[35] A. Turing. *On computable numbers, with an application to the Entscheidungsproblem* Proc. Lond. Math. Soc. 2, 42 230-265 (1936) 2, 5

[36] M. Viana. *Stochastic dynamics of deterministic systems* Brazillian Math. Colloquium 1997, IMPA. 14, 15

[37] L.S. Young. *What are SRB measures, and which dynamical systems have them?* Journal of Statistical Physics, Volume 108, Numbers 5-6, September 2002 , pp. 733-754(22) 2, 12

[38] K. Weihrauch. *A simple introduction to computable analysis.* Technical Report 171 7-1995, FernUniversitat, 1995. 9
[39] K. Weihrauch. *Computable Analysis*, Springer, 2000.M. 6